An Existential Crisis Resolved

Type inference for first-class existential types

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Despite the great success of inferring and programming with universal types, their dual—existential types—are much harder to work with. Existential types are useful in building abstract types, working with indexed types, and providing first-class support for refinement types. This paper, set in the context of Haskell, presents a bidirectional type-inference algorithm that infers where to introduce and eliminate existentials without any annotations in terms, along with an explicitly typed, type-safe core language usable as a compilation target. This approach is backward compatible. The key ingredient is to use strong existentials, which support (lazily) projecting out the encapsulated data, not weak existentials accessible only by pattern-matching.

Additional Key Words and Phrases: existential types, type inference, Haskell

1 INTRODUCTION

Parametric polymorphism through the use of universally quantified type variables is pervasive in functional programming. Given its overloaded numbers, a beginning Haskell programmer literally cannot ask for the type of 1 + 1 without seeing a universally quantified type variable.

However, universal quantification has a dual: existentials. While universals claim the spotlight, with support for automatic elimination (that is, instantiation) in all non-toy typed functional languages we know and automatic introduction (frequently, let-generalization) in some, existentials are underserved and impoverished. In every functional language we know, both elimination and introduction must be done explicitly every time, and languages otherwise renowned for their type inference—such as Haskell—require that users define a new top-level datatype for every existential.

While not as widely useful as universals, existential quantification comes up frequently in richly typed programming. Further examples are in Section 2, but consider writing a dropWhile function on everyone’s favorite example datatype, the length-indexed vector:

\[
\text{dropWhile} \quad \text{::} \quad (a \rightarrow \text{Bool}) \rightarrow \text{Vec} \ n \ a \rightarrow \text{Vec} \ ??? \ a
\]

How can we fill in the question marks? Without knowing the contents of the vector and the predicate we are passing, we cannot know the length of the output. Furthermore, returning an ordinary, unindexed list would requiring copying a suffix of the input vector, an unacceptable performance degradation.

Existentials come to our rescue: \[
\text{dropWhile} \quad \text{::} \quad (a \rightarrow \text{Bool}) \rightarrow \text{Vec} \ n \ a \rightarrow \exists m. \ Vec \ m \ a
\]. Though this example can be written today in a number of languages, all require annotations in terms both to pack (introduce) the existential and unpack (eliminate) it through the application or pattern-matching of a data constructor.

This paper describes a type-inference algorithm that supports implicit introduction and elimination of existentials, with a concrete setting in Haskell. We offer the following contributions:

• Section 4 presents our type-inference algorithm, the primary contribution of this paper. The algorithm is a small extension to an algorithm that accepts a Hindley-Milner language; our
language, $X$, is thus a superset of Hindley-Milner (Theorem 7.3). In addition, it supports several stability properties [Bottu and Eisenberg 2021]; a language is stable if small, seemingly innocuous changes to the input program (such as let-inlining) do not cause a change in the type or acceptability of a program (Theorems 7.4–7.6). Our algorithm is easily integrable with the latest inference algorithm [Serrano et al. 2020] in the Glasgow Haskell Compiler (GHC) (Section 8).

- Section 5 presents a core language based on System F, $FX$, that is a suitable target of compilation (Section 6) for $X$. We prove $FX$ is type-safe (Theorems 5.1 and 5.2) and supports type erasure (Theorem 5.3). It is designed in a way that is compatible with the existing System FC [Sulzmann et al. 2007] language used internally within GHC. All programs accepted by our algorithm elaborate to well-typed programs in $FX$ (Theorem 7.1). In addition, elaboration preserves the semantics of the source program, as we can observe by examining the result of type erasure (Theorem 7.2).

We normally desire type-inference algorithms to come with a declarative specification, where automatic introduction and elimination of quantifiers can happen anywhere, in the style of the Hindley-Milner type system [Hindley 1969; Milner 1978]. These specifications come alongside syntax-directed algorithms that are sound and complete with respect to the specification [Clément et al. 1986; Damas and Milner 1982]. However, we do not believe such a system is possible with existentials; while negative results are hard to prove conclusively, we lay out our arguments against this approach in Section 9.1. Instead, we present just our algorithm, though we avoid the complication and distraction of unification variables by allowing our algorithm to non-deterministically guess monotypes $\tau$ in the style of a declarative specification.

There is a good deal of literature in this area; much of it is focused on module systems, which often wish to hide the nature of a type using an existential package. We review some important prior work in Section 10.

The concrete examples in this paper are set in Haskell, but the fundamental ideas in our inference algorithm are fully portable to other settings, including in languages without let-generalization.

2 MOTIVATION AND EXAMPLES

Though not as prevalent as examples showing the benefits of universal polymorphism, easy existential polymorphism smooths out some of the wrinkles currently inherent in programming with indexed types such as GADTs [Xi et al. 2003].

2.1 Unknown Output Indices

We first return to the example from the introduction, writing an operation that drops an indeterminate number of elements from a length-indexed vector:

```haskell
data Nat = Zero | Succ Nat  
type Vec :: Nat -> Type -> Type  -- XStandaloneKindSignatures, new in GHC 8.10  
data Vec n a where  
    Nil :: Vec Zero a  
    (:) :: a -> Vec n a -> Vec (Succ n) a  

infixr 5 :>
```

In today’s Haskell, the way to write dropWhile over vectors is like this:

```haskell
type ExVec :: Type -> Type  
data ExVec a where  
    MkEV :: \(n::Nat\) (a::Type). Vec n a -> ExVec a
```

filter :: (a → Bool) → Vec n a → ExVec a
filter _ Nil = MkEV Nil
filter p (x ::> xs) | p x , MkEV v ← filter p xs = MkEV (x ::> v) | otherwise = filter p xs

(a)

(b)

Fig. 1. Implementations of filter over vectors (a) in today’s Haskell, and (b) with our extensions

dropWhile :: (a → Bool) → Vec n a → ExVec a
dropWhile _ Nil = MkEV Nil
dropWhile p (x ::> xs) | p x = dropWhile p xs | otherwise = x ::> dropWhile p xs
dropWhile _ Nil = Nil
dropWhile p (x ::> xs) | p x = x ::> dropWhile p xs | otherwise = dropWhile p xs

However, with our inference of existential introduction and elimination, we can simplify to this:

dropWhile :: (a → Bool) → Vec n a → ∃m. Vec m a
dropWhile _ Nil = Nil
dropWhile p (x ::> xs) | p x = dropWhile p xs | otherwise = x ::> xs

There are two key differences: we no longer need to define the ExVec type, instead using ⊤m. Vec m a; and we can omit any notion of packing in the body of dropWhile. Similarly, clients of dropWhile would not need to unpack the result, allowing the result of dropWhile to be immediately consumed by a map, for example.

2.2 Increased Laziness

Another function that produces an output of indeterminate length is filter. It is enlightening to compare the implementation of filter using today’s existentials and the version possible with our new ideas; see Figure 1.

Beyond just the change to the types and the disappearance of terms to pack and unpack existentials, we can observe that the laziness of the function has changed. (See Aside 1 for why we cannot easily make unpack bind lazily.) In Figure 1(a), we see that the recursive call to filter must be made before the use of the cons operator ::>. This means that, say, computing take 2 (filter p vec) (assuming take is clever enough to expect an ExVec) requires computing the result of the entire filter, even though the analogous expression on lists would only requiring filtering enough of vec to get the first two elements that satisfy p. The implementation of filter also requires enough stack space to store all the recursive calls, requiring an amount of space linear in the length of the input vector.

By contrast, the implementation in Figure 1(b) is lazy in the tail of the vector. Computing take 2 (filter p vec) really would only process enough elements of vec to find the first 2 that satisfy p. In addition, the computation requires only constant stack space, because filter will immediately return a cons cell storing a thunk for filtering the tail. If a bounded number of elements satisfy p, this is an asymptotic improvement in space requirements.

We can support the behavior evident in Figure 1(b) only because we use strong existential packages, where the existentially packed type can be projected out from the existential package,
What if `unpack` were simply lazy? The problem is that this is not simple! A straightforward typed operational semantics would not suffice, because there is no way to, say, reduce an `unpack` into a substitution (the way we would handle a lazy `let`). We could imagine an untyped operational semantics that did not require `unpack` to evaluate the existential package, binding its variable with a lazy binding. Without types, though, we would be unable to prove safety. In order to keep a typed operational semantics with a lazy `unpack`, we must model a set of heap bindings and an evaluation stack in our semantics. While this is possible, such an operational semantics is unsuitable for a (dependently typed) language where we also might wish to evaluate in types, which is our eventual goal for Haskell. The claim here is not that a lazy `unpack` is impossible, but that it is not obviously superior to the approach we advocate for here.

Relatedly, one could wonder whether we should just use a lazy Haskell pattern in `filter`. Alas, Haskell does not allow a lazy pattern to bind existential variables: writing `¬(MkEV v) « filter p xs` in Figure 1(a) would cause a compile-time error. This restriction in today’s Haskell is not incidental, because the internal language would require exactly the power of the open approach we propose here in order to support such a lazy pattern.

Aside 1. Why lazy `unpack` is no easy answer

instead of relying on the use of a pattern-match. Furthermore, projection of the packed type is requires no evaluation of any expression. We return to explain more about this key innovation in Section 3.

2.3 Object Encoding

Suppose we have a pretty-printer feature in our application, making use of the following class:

```haskell
class Pretty a where
  pretty :: a → Doc
```

There are `Pretty` instances defined for all relevant types. Now, suppose we have `order :: Order, client :: Client, and status :: OrderStatus`; we wish to create a message concatenating these three details. Today, we might say `vcat [ pretty order, pretty client, pretty status ]`, where `vcat :: [ Doc ] → Doc`. However, equipped with lightweight existentials, we could instead write `vcat [ order, client, status ]`, where `vcat :: [ ∃a. Pretty a ∧ a ] → Doc`. Here, the `∧` type constructor allows us to pack a witness for a constraint (such as a type class dictionary [Hall et al. 1996]) inside an existential package. Each element of the list is checked against the type ∃a. Pretty a ∧ a. Choosing one, checking `order` against ∃a. Pretty a ∧ a uses unification to determine that the choice of a should be Order, and we will then need to satisfy a `Pretty Order` constraint. In the implementation of `vcat`, elements of type ∃a. Pretty a ∧ a will be available as arguments to `pretty`:

```
vcat :: [ ∃a. Pretty a ∧ a ] → Doc
vcat [] = empty
vcat (x : xs) = pretty x $$ vcat xs
```

While the code simplification at call sites is modest, the ability to abstract over a constraint in forming a list makes it easier to avoid the types from preventing users from expressing their thoughts more directly.
Our main formal presentation in this paper does not include the packed constraints required here, but Section 9.2 considers an extension to our work that would support this example.

2.4 Richly Typed Data Structures

Suppose we wish to design a datatype whose inhabitants meet certain invariants by construction. If the invariants are complex enough, this can be done only by designing the datatype as a generalized algebraic datatype (GADT) [Xi et al. 2003]. Though other examples in this space abound (for example, encoding binary trees [McBride 2014] and regular expressions [Weirich 2018]), we will use the idea of a well-typed expression language, perhaps familiar to our readers.\(^1\)

The idea is encapsulated in these definitions:

```haskell
data Ty = Ty → Ty | ... -- base types elided
type Exp :: [Ty] → Ty → Ty → Type

data Exp ctx ty where
  App :: Exp ctx (arg → result) → Exp ctx arg → Exp ctx result
...
```

An expression of type `Exp ctx ty` is guaranteed to be well-typed in our object language: note that a function application requires the function to have a function type `arg → result` and the argument to have type `arg`. (The `ctx` is a list of the types of in-scope variables; using de Bruijn indices means we do not need to map names.) We are thus unable to represent the syntax tree applying, say, the number 5 to an argument `True`.

However, if we are to use `Exp` in a running interpreter, we have a problem: users might not type well-typed expressions. How can we take a user-written program and represent it in `Exp`? We must type-check it.

Assuming a type `UExp` (“unchecked expression”) that is like `Exp` but without its indices, we would write the following:\(^2\)

```haskell
typecheck :: (ctx :: [Ty]) → UExp → Maybe (∃ty. Exp ctx ty)
typecheck ctx (UApp fun arg) = do
  fun’ ← typecheck ctx fun
  arg’ ← typecheck ctx arg
  -- decompose the type of `fun’` into `expectedArgTy` :→ `resultTy`:
  (expectedArgTy, resultTy) ← checkFunctionTy (typeOf fun)

  -- Check whether `expectedArgTy` and the type of `arg’` are the same (failing if not)
  -- `Refl` is a proof the types coincide; matching on it reveals this fact to the type-checker:
  Refl ← checkEqual expectedArgTy (typeOf arg)
  return (App fun’ arg’)
```

The use of an existential type is critical here. There is no way to know what the type of an expression is before checking it, and yet we need this type available for compile-time reasoning to be able

\(^1\)This well-worn idea perhaps originates in a paper by Pfenning and Lee [1989], though that paper does not use an indexed datatype. Augustsson and Carlsson [1999] extend the idea to use a datatype, much as we have done here. A more in-depth treatment of this example is the subject of a functional pearl by Eisenberg [2020].

\(^2\)This rendering of the example assumes the ability to write using dependent types, to avoid clutter. However, do not be distracted: the dependent types could easily be encoded using singletons [Eisenberg and Weirich 2012; Monnier and Hagenauer 2010], while we focus here on the use of existntial types.
to accept the final use of \textit{App}. An example such as this one can be written today, but with extra awkward packing and unpacking of existentials, or through the use of a continuation-passing encoding. With the use of lightweight existentials, an example like this is easier to write, lowering the barrier to writing richly typed, finely specified programs.

3 KEY IDEA: EXISTENTIAL PROJECTIONS

In our envisioned source language, introduction and elimination of existential types are implicit. Precise locations are determined by type inference (as pinned down in Section 4)—accordingly, these locations may be hard to predict. Once these locations have been identified, the compiler must produce a fully annotated, typed core language that makes these introductions and eliminations explicit. We provide a precise account of this core language in Section 5. But before we do that, we use this section to informally justify why we need new forms in the first place. Why can we no longer use the existing encoding of existential types (based on Mitchell and Plotkin [1988] and Läufer [1996]) internally?

The key observation is that, since the locations of introductions and eliminations are hard to predict, they must not affect evaluation. Any other design would mean that programmers lose the ability to reason about when their expressions are reduced.

The existing datatype-based approach requires an existential-typed expression to be evaluated to head normal form to access the type packed in the existential. This is silly, however: types are completely erased, and yet this rule means that we must perform runtime evaluation simply to access an erased component of a some data.

To illustrate the problem, consider this Haskell datatype:

\begin{verbatim}
data Exists (f :: Type \rightarrow Type) = \forall (a :: Type). Ex ! (f a)
\end{verbatim}

With this construct, we can introduce existential types using the data constructor \textit{Ex} and eliminate them by pattern matching on \textit{Ex}. Note the presence of the strictness annotation, written with \texttt{!}. A use of the \textit{Ex} data constructor, if it is automatically inserted by the type inferencer, must not block reduction.\footnote{Similarly, our choice of explicit introduction form for the core language must be strict in its argument if it is to be unobservable.}

The difficult issue, however, is elimination. To access the value carried by \textit{Exists}, we must use pattern matching. We cannot use a straightforward projection function \texttt{unExists :: Exists f \rightarrow f \\ldots}: it would allow the abstracted type variable to escape its scope—exactly why we cannot write a well-scoped type signature for \texttt{unExists}. As a result, we cannot use this value without weak-head evaluation of the term. As Section 3.2 shows, this forcing can decrease the laziness of our program.

While perhaps not as fundamental as our desire for introduction and elimination to be transparent to evaluation, another design goal is to allow arbitrary \texttt{let}-inlining. In other words, if \texttt{let x = e1 in e2} type-checks, then \texttt{e2 [ e1 / x ]} should also type-check. This property gives flexibility to users: they (and their IDEs) can confidently refactor their program without fear of type errors.

Taken together, these design requirements—transparency to evaluation and support for \texttt{let}-inlining—drive us to enhance our core language with \textit{strong} existentials [Howard 1969]: existentials that allow projection of both the type witness and the packed value, without pattern-matching.\footnote{Strong existentials stand in contrast to \textit{weak} existentials. A strong existential package supports operators that access the encapsulated type and datum, while a weak existential requires pattern-matching in order to extract the datum and bring its type into scope. In a lazy language, strong existentials thus have greater expressive power, as we can use a lazy projection, as we do here.}
3.1 Strong Existentials via pack and open

Our core language \( \mathcal{F}_\pi \) adopts the following constructs for introducing and eliminating existential types:\(^5\)

\[
\text{Pack} \quad \Gamma \vdash e : \tau_2[\tau_1/a] \\
\Gamma \vdash \text{pack} \tau_1, e \text{ as } \exists a.\tau_2 : \exists a.\tau_2 \\
\text{Open} \quad \Gamma \vdash e : \exists a.\tau \\
\Gamma \vdash \text{open} e : \tau[\{ e : \exists a.\tau \}/a]
\]

The pack typing rule is fairly standard [Pierce 2002, Chapter 24]. This term creates an existential package, hiding a type \( \tau_1 \) in the package with an expression \( e \). Our operational semantics (Figure 7) includes a rule that makes this construct strict.

To eliminate existential types, we use the open construct (from Cardelli and Leroy [1990]) instead of pattern matching. The open construct eliminates an existential without forcing it, as opens are simply erased during compilation. The type of open \( e \) is interesting: we substitute away the bound variable \( a \), replacing it with \( \{ e : \exists a.\tau \} \). This type is an existential projection. The idea is that we can think of an existential package \( \exists a.\tau \) as a (dependent) pair, combining the choice for \( a \) (say, \( \tau_0 \)) with an expression of type \( \tau[\tau_0/a] \). The type \( \{ e : \exists a.\tau \} \) projects out the type \( \tau_0 \) from the pair.

A key aspect of open is that the type form \( \{ e : \exists a.\tau \} \) is a completely opaque type. In our surface language, \( \{ e : \exists a.\tau \} \) is equal to itself and no other type. Computation is not necessary in types. One way to think of this is to imagine that \( \{ e : \exists a.\tau \} \) is like a fresh type variable whose name is long—not as a construct that, say, accesses a type within \( e \).

The simple idea of open is very powerful. It means that we can talk about the type in an existential package without unpacking the package. It would even be valid to project out the type of an existential package that will never be computed. Because types can be erased in our semantics, even projecting out the type from a bottoming expression (of existential type) is harmless.\(^6\)

Note that the type of the existential package expression is included in the syntax for projections \( \{ e : \exists a.\tau \} \); this annotation is necessary because expressions in our surface language \( \mathcal{X} \) might have multiple, different types. (For example, \( \lambda x \rightarrow x \) has both type \( \text{Int} \rightarrow \text{Int} \) and type \( \text{Bool} \rightarrow \text{Bool} \).) Including the type annotation fixes our interpretation of \( e \), but see Section 6 for more on this point.

3.2 The unpack Trap

Adding the open term to the language comes at a cost to complexity. Let us take a moment to reflect on why a more traditional elimination form (called unpack) is insufficient.

A frequent presentation of existentials in a language based on System F uses the unpack primitive. Pierce [2002, Chapter 24] presents the idea with this typing rule:

\[
\text{Unpack} \\
\Gamma \vdash e_1 : \exists a.\tau_2 \\
\Gamma, a, x:\tau_2 \vdash e_2 : \tau \\
a \notin \text{fv}(\tau) \\
\Gamma \vdash \text{open} e_1 \text{ as } a, x \text{ in } e_2 : \tau
\]

The idea is that unpack extracts out the packed expression in a variable \( x \), also binding a type variable \( a \) to represent the hidden type. The typing rule corresponds to the pattern-match in case \( e_1 \) of Ex \( (x :: a) \rightarrow e_2 \), where \( x \) and \( a \) are brought into scope in \( e_2 \).\(^7\)

\(^5\)These rules are slightly simplified. The full rules appear in Section 5.

\(^6\)Readers may be alarmed at that sentence: how could \( \downarrow : \exists a.a \) be a valid type? Perhaps a more elaborate system might want to reject such a type, but there is no need to. As all types are erased and have no impact on evaluation, an exotic type like this is no threat to type safety.

\(^7\)See Eisenberg et al. [2018] for more details on how Haskell treats that type annotation.
This approach is attractive because it is simple to add to a language like System F. It does not require the presence of terms in types and the necessary machinery that we describe in Section 5. However, it is also not powerful enough to accommodate some of the examples we would like to support.

The unpack term impacts evaluation. Because it is based on pattern matching, the unpack term must reduce its argument to a weak-head normal form before providing access to the hidden type. The standard reduction rule looks like this:

\[
\text{unpack } (\text{pack } \tau_1, e_1 \text{ as } \exists a. \tau_2) \text{ as } a, x \text{ in } e_2 \rightarrow e_1 [e_1/x] [\tau_1/a]
\]

What this rule means is that the only parts of the term that have access to the abstract type are the ones that are evaluated after the existential has been weak-head normalized. Without weak-head normalizing the argument to a pack, we have nothing to substitute for \(x\) and \(a\).

Let us rewrite the filter example from Section 2.2, making more details explicit so that we can see why this is an issue.

Let us rewrite the filter example from Section 2.2, making more details explicit so that we can see why this is an issue.

\[
\text{filter} :: \forall n a. (a \rightarrow \text{Bool}) \rightarrow \text{Vec } n a \rightarrow \exists m. \text{Vec } m a
\]

\[
\text{filter } = \Lambda n a \rightarrow \lambda (p :: a \rightarrow \text{Bool}) (\text{vec} :: \text{Vec } n a) \rightarrow \\
\text{case vec of} \\
\quad (\cdot : ) n1 (x :: a) (xs :: \text{Vec } n1 a) \quad -- \text{vec is } x : > xs \\
\quad | p x \quad \rightarrow ... \\
\quad | \text{otherwise } \rightarrow \text{filter } n1 a p xs \\
\text{Nil } \rightarrow \text{pack } \text{Zero}, \text{Nil} \text{ as } \exists m. \text{Vec } m a \quad -- \text{vec is Nil}
\]

The treatment above makes all type abstraction and application explicit. Note that the pattern-match for the cons operator \( : \) includes a compile-time (or type-level) binding for the length of the tail, \(n1\).

The question here is: what do we put in the ... in the case where \(p x\) holds? One possibility is to apply the \((::)\) operator to build the result. However, right away, we are stymied: what do we pass to that operator as the length of the resulting vector? It depends on the length of the result of the recursive call. A use of unpack cannot help us here, as unpack is used in a term, not in a type index; even if we could use it, we would have to return the packed type, not something we can ordinarily do.

Instead, we must use unpack (and pack) before calling the \((::)\) operator. Specifically, we can write

\[
\text{unpack } \text{filter } n1 a p xs \text{ as } n2, ys \text{ in pack } n2, (\cdot : ) n2 x ys \text{ as } \exists m. \text{Vec } m a
\]

This use of unpack is type-correct, but we have lost the laziness of filter we so prized in Section 2.2.

On the other hand, open allows us to fill in the ... with the following code, using the the existential projection to access the new (type-level) length for the arguments to pack and to \(::\).

\[
\text{let } ys :: \exists m. \text{Vec } m a \quad -- \text{usual lazy let} \\
ys = \text{filter } n1 a p xs \\
\text{in pack } [ys :: \exists m. \text{Vec } m a], (\cdot : ) [ys :: \exists m. \text{Vec } m a] \times (\text{open } ys) \text{ as } \exists m. \text{Vec } m a
\]

As we expand on in the next subsection, we do not have to let-bind \(ys\); instead, we could just repeat the sub-expression filter \(n1 a p xs\).
3.3 The Importance of Strength

Beyond the peculiarities of the filter example, having a lazy construct that accesses the abstracted type in an existential package is essential to supporting inferrable existential types.

Here is a somewhat contrived example to illustrate this point:

```haskell
data Counter a = Counter { zero :: a, succ :: a → a, toInt :: a → Int }

mkCounter :: String → ∃ a. Counter a -- a counter with a hidden representation
mkCounter = ...

initial1 :: Int
initial1 = let c = mkCounter "hello" in (toInt c) (zero c)

initial2 :: Int
initial2 = (toInt (mkCounter "hello")) (zero (mkCounter "hello"))
```

We would like our language to accept both initial1 and initial2. After all, one of the benefits of working in a pure, lazy language is referential transparency: programmers (and tools, such as IDEs) should be able to perform expression inlining with no change in behavior. In both initial1 and initial2, the compiler must automatically eliminate the existential that results from each use of mkCounter. In the definition initial1, elaboration is not difficult, even if we only have the weak unpack elimination form to work with.

However, supporting initial2 is more problematic. Maintaining the order of evaluation of the source language requires two separate uses of the elimination form.

To type-check the application of toInt (mkCounter "hello") to zero (mkCounter "hello"), we must first know the type packed into the package returned from mkCounter "hello". Accessing this type should not evaluate mkCounter "hello", however: a programmer rightly expects that toInt is evaluated before any call to mkCounter is, which may have performance or termination implications. More generally, we can imagine the need for a hidden type arbitrarily far away from the call site of a function (such as mkCounter) that returns an existential; eager evaluation of the function would be most unexpected for programmers.

Note that, critically, both calls to mkCounter in initial2 contain the same argument. Since we are working in a pure context, we know that the result of the two calls to mkCounter "hello" in initial2 must be the same, and thus that the program is well-typed.

In sum, if the compiler is to produce the elimination form for existentials, that elimination form must be nonstrict, allowing the packed witness type to be accessed without evaluation. Any other choice means that programmers must expect hard-to-predict changes to the evaluation order of their program. In addition, if we wish to allow users to inline their let-bound identifiers, this projection form must also be strong, and remember the existentially typed expression in its type.

Note that we are taking advantage of Haskell’s purity in this part of the design. We can soundly support a strong elimination form like open only because we know that the expressions which appear in types are pure. All projections of the type witness from the same expression will be equal. In a language without this property, such as ML, we would need to enforce a value restriction on the type projections. Such a value restriction would prevent us from injecting, say, a non-deterministic expression into types; as there is no notion of evaluating a type, it would be unclear when and how often to evaluate the expression which could yield different results at each evaluation.

4 INFERRING EXISTENTIALS

In this section we present the surface language, X, that we use to manipulate existentials, and the bidirectional type system that infers them. As our concrete setting is in Haskell, our starting point...
is the surface language described by Serrano et al. [2020], modified to add support for existentials. We add a syntax for existential quantifiers $\exists a.\epsilon$ and existential projections $[e : \epsilon]$. An important part of our type system is the type instantiation mechanism, which implicitly handles the opening of existentials (Section 4.3).

### 4.1 Language Syntax
The syntax of our types is given in Figure 2.

$$
\sigma ::= \epsilon | \forall a.\sigma \quad \text{universally quantified type}
$$

$$
\epsilon ::= \rho | \exists b.\epsilon \quad \text{existentially quantified type}
$$

$$
\rho ::= \tau | \sigma_1 \rightarrow \sigma_2 \quad \text{top-level monomorphic type}
$$

$$
\tau ::= a | \text{Int} | \sigma_1 \rightarrow \sigma_2 | [e : \epsilon] \quad \text{monomorphic type}
$$

$$
a, b ::= \ldots \quad \text{type variable}
$$

$$
\Gamma ::= \emptyset | \Gamma, a | \Gamma, x:\sigma \quad \text{typing context}
$$

Fig. 2. Type stratification

Polytypes $\sigma$ can quantify an arbitrary number (including 0) universal variables and, within the universal quantification, an arbitrary number (including 0) existential variables. This stratification is enforced through the distinction between $\sigma$-types and $\epsilon$-types. Note that the type $\exists a.\forall b.\tau$ is ruled out.$^8$ Top-level monotypes $\rho$ have no top-level quantification. Monotypes $\tau$ include a projection form $[e : \epsilon]$ that occurs every time an existential is opened, as described in Section 3.1. Universal and existential variables draw from the same set of variable names, denoted with $a$ or $b$.

The expressions of $\mathcal{X}$ are defined as follows:

$$
x ::= \ldots \quad \text{term variable}
$$

$$
n ::= \ldots \quad \text{integer literal}
$$

$$
e ::= h\pi | \lambda x.e | \text{let } x = e_1 \text{ in } e_2 | n \quad \text{expression}
$$

$$
h ::= x | e | e :: \sigma \quad \text{expression head}
$$

$$
\pi ::= e | \sigma \quad \text{argument}
$$

Fig. 3. Our surface language, $\mathcal{X}$

This language is a fairly small $\lambda$-calculus, with type annotations and $n$-ary application (including type application). The expression $h\pi_1 \ldots \pi_n$ applies a head to a sequence of arguments $\pi_i$ that can be expressions or types. The head is either a variable $x$, an annotated expression $e :: \sigma$, or an expression $e$ that is not an application.$^9$

An important complication of our type system is that expressions may appear in types: this happens in the projection form $[e : \epsilon]$. We thus must address how to treat type equality. For example, suppose term variable $x$ (of type $\epsilon$) is free in a type $\tau$; is $\tau[(\lambda y.y) 1 / x]$ equal to $\tau[1 / x]$?

---

$^8$As usual, stratifying the grammar of types simplifies type inference. In our case, this choice drastically simplifies the challenge of comparing types with mixed quantifiers. Dunfield and Krishnaswami [2019, Section 2] have an in-depth discussion of this challenge.

$^9$Our grammar does not force a head expression $h$ to be something other than an application, but we will consistently assume this restriction is in force. It would add clutter and obscure our point to bake this restriction in the grammar.
\[
\text{(Universal type checking)}
\]
\[
\begin{align*}
\Gamma \vdash^v e & \Leftrightarrow \sigma \\
\text{Gen} & \\
\Gamma, \bar{\alpha} \vdash e & \Leftrightarrow \rho[\overline{\tau} / \overline{b}] \\
\text{fv}(\overline{\tau}) & \subseteq \text{dom}(\Gamma, \bar{\alpha}) \\
\Gamma \vdash^v e & \Leftrightarrow \forall \bar{\alpha}, \exists \overline{b}, \rho
\end{align*}
\]
\[
\text{(Type synthesis and type checking)}
\]
\[
\begin{align*}
\Gamma \vdash e & \Rightarrow \rho & \Gamma \vdash e & \Leftrightarrow \rho \\
\text{App} & \\
\Gamma \vdash_h h \Rightarrow \sigma & \\
\text{H-Var} & \\
x: \sigma \in \Gamma & \\
\Gamma \vdash_h x \Rightarrow \sigma & \Gamma \vdash_h (e :: \sigma) \Rightarrow \sigma \\
\Gamma \vdash_h e \Rightarrow \rho & \Gamma \vdash_h e \Rightarrow \rho
\end{align*}
\]

Fig. 4. Type inference for X

That is, does type equality respect \(\beta\)-reduction? Our answer is “no”: we restrict type equality in our language to be syntactic equality (modulo \(\alpha\)-equivalence, as usual). We can imagine a richer type equality relation—which would accept more programs—but this simplest, least expressive version satisfies our needs. (However, see Aside 2 in Section 7.3 for a wrinkle here.) Adding such an equality relation is largely orthogonal to the concerns around existential types that draw our focus.\(^{10}\)

4.2 Type System

The typing rules of our language appear in Figure 4. This bidirectional type system uses two forms for typing judgments: \(\Gamma \vdash e \Rightarrow \rho\) means that, in the type environment \(\Gamma\), the program \(e\) has the inferred type \(\rho\), while \(\Gamma \vdash e \Leftrightarrow \rho\) means that, in the type environment \(\Gamma\), \(e\) is checked to have type \(\rho\). We also use a third form to simplify the presentation of the rules: \(\Gamma \vdash e \Rightarrow \rho\) means that the rule can be read by replacing \(\Rightarrow\) with either \(\Rightarrow\) or \(\Leftrightarrow\) in both the conclusion and premises. Although the rules are fairly close to the standard rules of a typed \(\lambda\)-calculus, handling existentials through packing and opening has an impact on the rules \text{LET} and \text{GEN}.

\(^{10}\)Our core language FX does need to think harder about this question, in order to prove type safety. See Section 5.1.
We review the rules in Figure 4 here, deferring the most involved rule, APP, until after we discuss the instantiation judgment \( \Gamma \vdash \text{inst} \), in Section 4.3.

### 4.2.1 Simple Subsumption

Bidirectional type systems typically rely on a reflexive, transitive subsumption relation \( \leq \), where we expect that if \( e : \sigma_1 \) and \( \sigma_1 \leq \sigma_2 \), then \( e : \sigma_2 \) is also derivable. For example, we would expect that \( \forall \, a.\, a \rightarrow a \leq \text{Int} \rightarrow \text{Int} \). This subsumption relation is then used when "switching modes"; that is, if we are checking an expression \( e \) against a type \( \sigma_2 \) where \( e \) has a form resistant to type propagation (the case when \( e \) is a function call), we infer a type \( \sigma_1 \) for \( e \) and then check that \( \sigma_1 \leq \sigma_2 \).

However, our type system refers to no such \( \leq \) relation: we essentially use equality as our subsumption relation, invoking it implicitly in our rules through the use of a repeated metavariable. (Though hard to see, the repeated metavariable is the \( \rho_r \) in rule APP, when replacing the \( \equiv \) in the conclusion with \( a \equiv \).) We get away with this because our bidirectional type-checking algorithm works over top-level monotypes \( \rho \), not the more general polytype \( \sigma \). A type \( \rho \) has no top-level quantification at all. Furthermore, our type system treats all types as invariant—including \( \rightarrow \). This treatment follows on from the ideas in Serrano et al. [2020, Section 5.8], which describes how Haskell recently made its arrow type similarly invariant.

We adopt this simpler approach toward subsumption both to connect our presentation with the state-of-the-art for type inference in Haskell [Serrano et al. 2020] and also because this approach simplifies our typing rules. We see no obstacle to incorporating our ideas with a more powerful subsumption judgment, such as the deep-skolemization judgment of Peyton Jones et al. [2007, Section 4.6.2] or the slightly simpler co- and contravariant judgment of Odersky Jones et al. [1996, Figure 2].

### 4.2.2 Checking against a Polytype

Rule GEN, the sole rule for the \( \Gamma \vdash \forall \, e \equiv \sigma \) judgment, deals with the case when we are checking against a polytype \( \sigma \). If we want to ensure that \( e \) has type \( \sigma \), then we must skolemize any universal variables bound in \( \sigma \): these variables behave essentially as fresh constants while type-checking \( e \). Rule GEN thus just brings them into scope.

On the other hand, if there are existential variables bound in \( \sigma \), then we must instantiate these. If we are checking that \( e \) has some type \( \exists \, a.\, \tau_0 \), that means we must find some type \( \tau \) such that \( e \) has type \( \tau_0[\tau / a] \). This is very different than the skolemization of a universal variable, where we must keep the variable abstract. Instead, when checking against \( \exists \, a.\, e \), we guess a monotype \( \tau \) and check \( e \) against the type \( e[\tau / a] \). Rule GEN simply does this for nested existential quantification over variables \( \bar{b} \). A real implementation might use unification variables, but we here rely on the rich body of literature [e.g., Dunfield and Krishnaswami 2013] that allows us to guess monotypes during type inference, knowing how to translate this convention into an implementation using unification variables.

### 4.2.3 Abstractions

Rule IABS synthesizes the type of a \( \lambda \)-abstraction, by guessing the (mono)type \( \tau \) of the bound variable and then inferring the type of the body \( e \) to be \( \rho \). However, rule IABS also can pack existentials. This is necessary to avoid skolem escape: it is possible that the type \( \rho \) contains \( x \) free. However, it would be disastrous if \( \lambda \, x.\, e \) was assigned a type mentioning \( x \), as \( x \) is no longer in scope.

For example, suppose we have \( \Gamma = f:\text{Int} \rightarrow \exists \, a.\, a \rightarrow \text{Bool} \). Now, consider inferring the type \( \rho \) in \( \Gamma \vdash \lambda \, x.\, f \, x \equiv \rho \). Guessing \( x : \text{Int} \), we will infer \( \Gamma, x : \text{Int} \vdash f \, x \equiv [f \, x : \exists \, a.\, a \rightarrow \text{Bool}] \rightarrow \text{Bool} \). It is tempting, then, to say \( \Gamma \vdash \lambda \, x.\, f \, x \equiv \text{Int} \rightarrow [f \, x : \exists \, a.\, a \rightarrow \text{Bool}] \rightarrow \text{Bool} \), but this is wrong: the type mentions \( x \) free, but \( \Gamma \) does not bind \( x \). Instead, rule IABS infers \( \Gamma \vdash \lambda \, x.\, f \, x \equiv \exists \, a.\, \text{Int} \rightarrow a \rightarrow \text{Bool} \), by using \( a \) instead of the ill-scoped \( [f \, x : \exists \, a.\, a \rightarrow \text{Bool}] \).
More generally, we must identify all existential projections within $\rho$ that have $x$ free. These are replaced with fresh variables $\tilde{a}$. We use the notation $[\rho]_{\tilde{a}}$ to denote the list of projections in $\rho$; multiple projections of the same expression (that is, multiple occurrences of $[e_0 : e_0]$ for some $e_0$ and $e_0$) are commoned up in this list. Formally,

$$[\rho]_x = \{[e : e] \mid ([e : e] \text{ is a sub-expression of } \rho) \land (x \text{ is a free variable in } e)\}.$$  

The notation $\rho[\tilde{a}] / [\rho]_{\tilde{a}}$ denotes the type $\rho$ where the $\tilde{a}$ are written in place of these projections. Note that this notation is set up backward from the way it usually works, where we substitute some type for a variable. Here, instead, we are replacing the type with a fresh variable.

In the conclusion of the rule, we existentially quantify the $\tilde{a}$, to finally obtain a function type of the form $\tau \rightarrow \exists \tilde{a}, \rho$.

The checking rule \texttt{CABS} is much simpler. We know the type of the bound variable by decomposing the known expected type $\sigma_1 \rightarrow \sigma_2$. We also need not worry about skolem escape because we have been provided with a well-scoped $\sigma_2$ result type for our function. The only small wrinkle is the need to use $\iota^V$ in order to invoke rule \texttt{GEN} to remove any quantifiers on the type $\sigma_2$.

### 4.2.4 Let Skolem-escape

Rule \texttt{LET} deals with \texttt{let}-expressions, both in synthesis and in checking modes. It performs standard \texttt{let}-generalization, computing generalized variables $\tilde{a}$ by finding the free variables in $\rho_1$ and removing any variables additionally free in $\Gamma$. Indeed, all that is unexpected in this rule is the type substitution in the conclusion.

The problem, like with rule \texttt{IABS} is the potential for skolem-escape. The variable $x$ might appear in the type $\rho_2$. However, $x$ is out of scope in the conclusion, and thus it cannot appear in the overall type of the \texttt{let}-expression. One solution to this problem would be to pack all the existentials that fall out of scope, much like we do in rule \texttt{IABS}. However, doing so would mean that our bidirectional type system now infers existential types $e$ instead of top-level monomorphic types $\rho$; keeping with the simpler $\rho$ is important to avoid the complications of a non-trivial subsumption judgment. Hence we choose to replace all occurrences of $x$ inside of projections by the expression $e_1$. This does not pose a problem since $e_1$ is well-typed according to the premises of the \texttt{LET} rule.

### 4.2.5 Inferring the Types of Heads

Following Serrano et al. [2020], our system treats $n$-ary applications directly, instead of recurring down a chain of binary applications $e_1, e_2$. The head of an $n$-ary application is denoted with $h$; heads’ types are inferred with the $\Gamma \vdash h \Rightarrow \sigma$ judgment. Variables simply perform a context lookup, annotated expressions check the contained expression against the provided type, and other expressions infer a $\rho$-type. It is understood here that we use rule \texttt{H-INFER} only when the other rules do not apply, for example, for $\lambda$-abstractions.

### 4.3 Instantiation Semantics

The instantiation rules of Figure 5 present an auxiliary judgment used in type-checking applications. The judgment $\Gamma \vdash^{\text{inst}} e : \sigma ; \bar{\pi} \sim \sigma ; \rho_r$ means: with in-scope variables $\Gamma$, apply function $e$ of type $\sigma$ to arguments $\bar{\pi}$ requires $\text{exprargs}(\bar{\pi})$ (the value arguments) to have types $\bar{\sigma}$, resulting in an expression $e\bar{\pi}$ of type $\rho_r$. This judgment is directly inspired by Serrano et al. [2020, Figure 4].

The idea is that we use $\iota^{\text{inst}}$ to figure out the types of term-level arguments to a function in a pre-pass that examines only type arguments. Having determined the expected types of the term-level arguments $\bar{\sigma}$, rule \texttt{APP} (in Figure 4) actually checks that the arguments have the correct types. This pre-pass is not necessary in order to infer the types for existentials, but it sets the stage for Section 8, where we integrate our design with the current implementation in GHC.

---

11Our language works well without this special substitution. Instead, we could have a check that the final inferred type in rule \texttt{IABS} is well scoped. However, this extra existential packing is easy enough to add, and so we have.
Application. Rule ITYARG handles type application by instantiating the bound variable \( a \) with the supplied type argument \( \sigma' \). Rule IARG handles routine expression application simply by remembering that the argument should have type \( \sigma_1 \). Note that we do not check that the argument \( e' \) has type \( \sigma_1 \) here.

Quantifiers. Rule IALL deals with universal quantifiers in the function’s type by instantiating with a guessed monotype \( \tau \). The first premise is to avoid ambiguity with rule ITYARG; we do not wish to guess an instantiation when the user provides it explicitly with a type argument.

Rule IEXIST eagerly opens existentials by substituting a projection in place of the bound variable \( a \). This is the only place in the judgment where we need the function expression \( e \); whenever we open an existential type, we must remember what expression has that type, so that we do not confuse two different existentially packed types.

For example, if \( f \) has type \( \text{Bool} \rightarrow \exists b. (b, b \rightarrow \text{Int}) \), then the function application \( f \, \text{True} \) will be given the opened pair type:

\[
(\lfloor f \, \text{True} : \exists b. (b, b \rightarrow \text{Int}) \rfloor, \lfloor f \, \text{True} : \exists b. (b, b \rightarrow \text{Int}) \rfloor) \rightarrow \text{Int}
\]

Rule IRESULT concludes computing the instantiation in a function application by copying the function type to be the result type.

The App rule. Having now understood the instantiation judgment, we turn our attention to rule APP. After inferring the type \( \sigma \) for an application head \( h \), \( \sigma \) gets instantiated, revealing argument types \( \overline{\sigma} \). Each argument \( e_i \) is checked against its corresponding type \( \sigma_i \), where the entire function application expression has type \( \rho_r \). Rule APP operates in both synthesis and checking modes. When synthesizing, it simply returns \( \rho_r \) from the instantiation judgment; when checking, it ensures that the instantiated type \( \rho_r \) matches what was expected. We need do no further instantiation or skolemization because we have a simple subsumption relation.

5 CORE LANGUAGE

Perhaps we can infer existential types using existential projections \( \lfloor e : e \rfloor \), but how do we know such an approach is sound? We show that it is by elaborating our surface expressions into a core
language $\mathcal{FX}$, inspired by a similar language described by Cardelli and Leroy [1990, Section 4], and we prove the standard progress and preservation theorems of this language. This section presents $\mathcal{FX}$ and states key metatheory results; the following section connects $\mathcal{X}$ to $\mathcal{FX}$ by presenting our elaboration algorithm.

The syntax of $\mathcal{FX}$ is in Figure 6 and selected typing rules are in Figure 7; full typing rules appear in the appendix.\footnote{https://richarde.dev/papers/2021/exists/exists-extended.pdf} Note that we use upright Latin letters to denote $\mathcal{FX}$ expressions and types; when we mix $\mathcal{X}$ and $\mathcal{FX}$ in close proximity, we additionally use colors.

The nub of $\mathcal{FX}$ is System F, with fully applied base types $\mathcal{B}$ (because they are fully applied, we do not need to have a kind system) and ordinary universal quantification. We thus omit typing rules from this presentation that are standard. The inclusion of existential types, $\text{pack}$ and $\text{open}$ is fitting for a core language supporting existentials. This language necessarily has mutually recursive grammars for types and expressions, but the typing rules are not mutually recursive: rule $\text{CT-PROJ}$ shows that a projection in a type is well-formed when the expression is well-scoped. (The $\vdash \text{G ok}$ premise refers to a routine context-well-formedness judgment, omitted.) We do not require the existential package to be well-typed (though it would be, in practice).

### 5.1 Coercions

The biggest surprise in $\mathcal{FX}$ is its need for type and expression coercions. The motivation for these can be seen in rule $\text{CS-OPENPACK}$. If we are stepping an expression $\text{open}$ ($\text{pack } t, v \text{ as } \exists a. t_2$), we want to extract the value $v$ from the existential package. The problem is that $v$ has the wrong type. Suppose that $v$ has type $t_0$. Then, we have $\text{pack } t, v \text{ as } \exists a. t_2 : \exists a. t_2$ and $\text{open}$ ($\text{pack } t, v \text{ as } \exists a. t_2$) $:\text{t2}[,\text{pack t, v as } \exists a. t_2] / a]$, according to rule $\text{CE-OPEN}$. This last type is not syntactically the same as $t_0$, although it must be that $t_0 = t_2[t / a]$ to satisfy the premises of rule $\text{CE-PACK}$. Because the type of the opened existential does not match the type of the packed value, a naive reduction rule like $\text{G \vdash open (pack t, v as t2) } \rightarrow v$ would not preserve types.

There are, in general, two ways to build a type system when encountering such a problem. We could have a non-trivial type equality relation, where we say that $[\text{pack t, e as t2}] \equiv t$. Doing so would simplify the reduction rules, but this simplification comes at a cost: our language would now have a conversion rule that allows an expression of one type $t_1$ to have another type $t_2$ as long as $t_1 \equiv t_2$. This rule is not syntax-directed; accordingly, it is hard to determine whether type-checking remains decidable. Furthermore, a non-trivial type equality relation makes proofs considerably more involved. In effect, we are just moving the complexity we see in the right-hand side of a rule like rule $\text{CS-OPENPACK}$ into the proofs.
\[ G \vdash e : t \]

**Expression typing**

- **CE-Abs**
  \[ G, x : t_1 \vdash e : t_2 \]
  \[ x \not\in \text{fv}(t_2) \]
  \[ G \vdash \lambda x : t_1. e : t_2 \]

- **CE-Let**
  \[ G, x_1 : t_1 \vdash e_1 : t_1 \]
  \[ G, x_2 : t_2 \vdash e_2 : t_2 \]
  \[ G \vdash \text{let } x_1 = e_1 \text{ in } e_2 \vdash t_2[x_1 / x] \]

- **CE-Open**
  \[ G \vdash e : \exists t \]
  \[ G \vdash \text{open } e : t \]

- **CE-Cast**
  \[ G \vdash e : t \]
  \[ G \vdash \gamma : t \sim t_2 \]
  \[ G \vdash e \rightarrow \gamma : t_2 \]

**Type well-formedness**

- **CG-Refl**
  \[ G \vdash t : \text{type} \]
  \[ G \vdash t_1 \sim t_2 \]

- **CG-Sym**
  \[ G \vdash \gamma : t_1 \sim t_2 \]
  \[ G \vdash \text{sym } \gamma : t_2 \sim t_1 \]

- **CG-InstExists**
  \[ G \vdash \gamma_1 : (\exists a). t_1 \sim (\exists a). t_2 \]
  \[ G \vdash \gamma_2 : t_3 \sim t_4 \]
  \[ G \vdash \gamma_1 \circ \gamma_2 : t_1[t_3 / a] \sim t_2[t_4 / a] \]

- **CG-Coherence**
  \[ G \vdash e : t_1 \]
  \[ G \vdash \gamma : t_1 \sim t_2 \]
  \[ G \vdash e \rightarrow \gamma : (e \rightarrow \gamma) \]

**Type coercion typing**

- **CG-Trans**
  \[ G \vdash \gamma_1 : t_1 \sim t_2 \]
  \[ G \vdash \gamma_2 : t_2 \sim t_3 \]
  \[ G \vdash \gamma_1 \circ \gamma_2 : t_1 \sim t_3 \]

- **CG-Proj**
  \[ G \vdash \eta : e_1 \sim e_2 \]

- **CG-ProjPack**
  \[ G \vdash \text{pack } t, e \text{ as } t_2 : t_2 \]
  \[ G \vdash \text{projpack } t, e \text{ as } t_2 : [\text{pack } t, e \text{ as } t_2] \sim t \]

**Small-step operational semantics**

- **CH-Step**
  \[ G \vdash e : t \]
  \[ G \vdash e' : t \]
  \[ G \vdash e \rightarrow e' \]

- **CS-PackCong**
  \[ G \vdash e \rightarrow e' \]

- **CS-OpenPack**
  \[ G \vdash \text{open } (\text{pack } t, v \text{ as } t_2) \rightarrow v \rightarrow \langle t_2 \rangle \circ (\text{sym } (\text{projpack } t, v \text{ as } t_2)) \]

- **CS-OpenCong**
  \[ G \vdash e : t \]
  \[ G \vdash e \rightarrow e' \]
  \[ G \vdash \text{open } e \rightarrow \text{open } e' \rightarrow (t) \circ (\text{sym } (\text{step } e)) \]

Fig. 7. Selected typing and reduction rules of the core language, FX

The alternative approach to a non-trivial equality relation is to use explicit coercions, as we have here. The cost is clutter. Casts sully our reduction steps, and we need to explicitly shunt coercions in several (omitted, unenlightening) reduction rules— for example, when reducing \(((\lambda x : t . e_1) \triangleright \gamma) e_2\) where the cast intervenes between a \(\lambda\)-abstraction and its argument. Despite the presence of these rules in our operational semantics, coercions can be fully erased: we can write an alternative, untyped operational semantics that omits coercions entirely. Theorem 7.2 shows that erasure preserves program behavior.

Both approaches—an enriched definitional equality vs. explicit coercions—are essentially equivalent: we can view explicit coercions simply as an encoding of the derivation of an equality judgment.\(^{13}\) We choose explicit coercions both because \(FX\) is a purely internal language (and thus clutter is less noisome) and because it allows for an easy connection to the implementation of the core language in GHC, based on System FC [Sulzmann et al. 2007], with similar explicit coercions.

The coercion language for \(FX\) includes constructors witnessing that they encode an equivalence relation (rules CG-Refl, CG-Sym, and CG-Trans), along with several omitted forms showing that the equivalence is also a congruence over types. Coercions also include several decomposition operations; rule CG-InstExists shows one, used in our reduction rules. The two forms of interest to use are \([\eta]\) (rule CG-Proj) and projpack (rule CG-ProjPack). The former injects the equivalence relation on expressions (witnessed by expression coercions \(\eta\)) into the type equivalence relation, and the latter witnesses the equivalence between \([\text{pack} t, e \ as \ t]\) and its packed type \(t\).

The equivalence relation on expressions is surprisingly simple: we need only the two rules in Figure 7. These rules allow us to drop casts (supporting a coherence property which states that the presence of casts is essentially unimportant) and to reduce expressions.

5.2 Metatheory

We prove (almost) standard progress and preservation theorems for this language:

**Theorem 5.1 (Progress).** If \(G \vdash e : t\), where \(G\) contains only type variable bindings, then one of the following is true:

1. there exists \(e'\) such that \(G \vdash e \rightarrow e'\);
2. \(e\) is a value \(v\); or
3. \(e\) is a casted value \(v \triangleright \gamma\).

**Theorem 5.2 (Preservation).** If \(G \vdash e : t\) and \(G \vdash e \rightarrow e'\), then \(G \vdash e' : t\).

In addition, we prove that types can still be erased in this language. Let \(|e|\) denote the expression \(e\) with all type abstractions, type applications, packs, opens and casts dropped. Furthermore, overload \(\rightarrow\) to mean the reduction relation over the erased language.

**Theorem 5.3 (Erasure).** If \(G \vdash e \rightarrow e'\), then \(|e| \rightarrow |e'|\).

The proofs largely follow the pattern set by previous papers on languages with explicit coercions and are unenlightening. They appear, in full, in the appendix.

6 ELABORATION

We now augment our inference rules from Section 4 to describe the elaboration from the surface language \(X\) into our core \(FX\). The notation \(\Rightarrow\) denotes elaboration of a surface term, type or context into its core equivalent. Some of our rules appear in Figure 8. The rest appear in the appendix. In order to aid understanding, we use blue for \(X\) terms and red for \(FX\) terms.

\(^{13}\)Weirich et al. [2017] makes this equivalence even clearer by presenting two proved-equivalent versions of a language, one with a non-trivial, undecidable type equality relation and another with explicit coercions.
Elab-Gen

\[
\frac{\Delta, \vec{a} \vdash e \iff \rho[\vec{a} / \vec{b}] \Rightarrow e}{\Gamma \vdash e \iff \forall \vec{a}. \exists \vec{b}. \rho \Rightarrow \Lambda \vec{a}. \text{pack}\ ] e \text{ as } \exists \vec{b}. \rho}
\]

Elab-iAbs

\[
\begin{align*}
\vec{a} & \text{ fresh} \\
\Delta, x : \tau \vdash e & \Rightarrow \rho \Rightarrow e \\
fv(\tau) & \subseteq \text{dom}(\Gamma) \\
\rho' & = \rho[\vec{a} / [\rho],.] \\
\tau & \Rightarrow t \\
\rho & \Rightarrow r \
\rho' & \Rightarrow r'
\end{align*}
\]

Elab-App

\[
\frac{\Gamma \vdash h \Rightarrow \sigma \Rightarrow h}{\Gamma \vdash \text{inst} \ h : \sigma \Rightarrow h ; \bar{\pi} \Rightarrow \bar{\sigma} ; \rho_r \Rightarrow e_r}
\]

Elab-IArg

\[
\frac{\Gamma \vdash \text{inst} \ e e' : \sigma_2 \Rightarrow e e' ; \bar{\pi} \Rightarrow \bar{\sigma} ; \rho_r \Rightarrow e_r}{\Gamma \vdash \text{inst} \ e : (\sigma_1 \rightarrow \sigma_2) \Rightarrow e ; \bar{\pi} \Rightarrow \sigma_1, \bar{\sigma} ; \rho_r \Rightarrow e_r}
\]

Elab-IExist

\[
\frac{\Gamma \vdash \text{inst} \ e e[a] \Rightarrow e[a] ; \bar{\pi} \Rightarrow \bar{\sigma} ; \rho_r \Rightarrow e_r}{\Gamma \vdash \text{inst} \ e : \exists a.e \Rightarrow e ; \bar{\pi} \Rightarrow \bar{\sigma} ; \rho_r \Rightarrow e_r}
\]

Elab-IResult

\[
\frac{\Gamma \vdash \text{inst} \ e \Rightarrow e_r \ ; [a] \sim [a] ; \rho_r \Rightarrow e_r}{\Gamma \vdash \text{inst} \ e : \rho_r \Rightarrow e_r}
\]

Fig. 8. Judgments and selected rules for elaborating from \(X\) into \(FX\).

The rules in Figure 8 allow packing multiple existentials at once, when given a list of types as the first argument to \texttt{pack}; see rules \texttt{Elab-Gen} and \texttt{Elab-iAbs}. Rule \texttt{Elab-Gen} checks a surface expression \(e\) against an expected type \(\forall \vec{a}. \exists \vec{b}. \rho\). We see that the result of elaboration uses nested \(\Lambda\)-abstractions and our nested \texttt{pack} notation to produce an \(FX\) expression that has the desired type. Rule \texttt{Elab-iAbs} echoes rule \texttt{iAbs}, producing an \(FX\) expression with \texttt{packs} necessary to accommodate any projections that mention the bound variable \(x\); recall the special treatment of such projections from Section 4.2.3.

Rule \texttt{Elab-App} elaborates the head \(h\) to \(h\), and then calls the \texttt{inst} judgment. This judgment takes the elaborated \(h\) as an \texttt{input} (despite its appearance on the right of \(\Rightarrow\)). This input of an elaborated expression is built up as the application spine is checked, to be returned in rule \texttt{Elab-IResult}.

In order to build this elaborated expression as we go, rule \texttt{Elab-IArg} elaborates arguments, in...
contrast to our original rule \texttt{IARG}; rule \texttt{ELAB-APP} then no longer needs to check these arguments in a second pass.\footnote{Knowledgeable readers will wonder how this new treatment interacts with the Quick-Look algorithm, which critically depends on waiting to type-check arguments after a quick look at the entire argument spine. The solution is to be lazy: the elaborated is not needed until after all arguments have been checked. Accordingly, we could, for example, use a mutable cell to hold the elaborated expression, and then fill in this cell only during the second pass. Our formal presentation here need not worry about this technicality, however.} Rule \texttt{ELAB-IEXIST} is the place where \texttt{open} is introduced, as it open an expression with an existential type.

The omitted rules all appear in the appendix and broadly follow the pattern set here.

### 6.1 Tweaking the IExist Rule

In the instantiation judgment for the surface language (Figure 5), rule \texttt{IEXIST} opens existentials. That is, given an expression \( e \) with an existential type \( \exists a.e \), it infers for \( e \) the type resulting from replacing the type variable with the projection \([ e : \exists a.e ]\). However, these projections pose a problem during the elaboration process. Specifically, if we have an application \( e_1 e_2 \) such that \( e_1 \) expects an argument whose type mentions \([ e_0 : e ]\)—and \( e_2 \) indeed has a type mentioning \([ e_0 : e ]\)—we cannot be sure that the application remains well-typed after elaboration. After all, type-checking in \( X \) is non-deterministic, given the way it guesses instantiations and the types of \( \lambda \)-bound variables. Another wrinkle is that \([ e_0 : e ]\) might appear under binders, making it even easier for type inference to come to two different conclusions when computing \( \Gamma \vdash e_0 \Leftarrow e \).

There are two approaches to fix this problem: we can require our elaboration process to be deterministic, or we can modify rule \texttt{IEXIST} to make sure that projections in the surface language actually use pre-elaborated core expressions. We take the latter approach, as it is simpler and more direct. However, we discuss later in this section the possible disadvantages of this choice, and a route to consider the first one.

Accordingly, we now introduce the following new \texttt{IEXISTCore} and rule \texttt{LetCore} rules, replacing rules \texttt{IEXIST} and rule \texttt{LET}:

\[
\begin{align*}
\text{IEXISTCore} \quad & \quad \Gamma \vdash^\forall e \Leftarrow \exists a.e \Rightarrow e \\
\text{LetCore} \quad & \quad \Gamma \vdash e_1 \Rightarrow \rho_1 \Leftarrow e_1 \\
& \quad \Gamma \vdash e[\{ e \} / a] ; \pi \leadsto \sigma ; \rho_r \\
& \quad \Gamma \vdash^\text{inst} e : \exists a.e ; \pi \leadsto \sigma ; \rho_r \\
& \quad \Gamma \vdash let x = e_1 in e_2 \Rightarrow \rho_2 [\Lambda a.e_1 / x]
\end{align*}
\]

Fig. 9. Updated rules to support \texttt{FX} expressions in \( X \) types

Now, the elaboration process \( \tau \Rightarrow t \) is indeed deterministic, making \( \Rightarrow \) a function on types \( \tau \) and contexts \( \Gamma \). Having surmounted this hurdle, elaboration largely very straightforward.

### 6.2 A Different Approach

We may want to refrain from using core expressions inside of projections, because doing so introduces complexity for the programmer who is not otherwise exposed to the core language. To wit, \( X \) would keep using projections of the form \([ e : e ]\), where we understand that \( \Gamma \vdash^\forall e \Leftarrow e \) in the ambient context \( \Gamma \), while \texttt{FX} uses the form \([ e ]\).

It is vitally important that, if our surface-language typing rules accept a program, the elaborated version of that program is type-correct. (We call this property \textit{soundness}; it is Theorem 7.1.) Yet, if elaboration of types is non-deterministic, we will lose this property, as explained above.
This alternative approach is simply to assume that elaboration is deterministic. Doing so is warranted because, in practice, a type-checker implementation will proceed deterministically—it seems far-fetched to think that a real type-checker would choose different types for the same expression and expected type, if any. In essence, a deterministic elaborator means that we can consider $\lfloor e : \epsilon \rfloor$ as a proxy for $\lfloor e \rfloor$. The first is preferable to programmers because it is written in the language they program in. However, a type-checker implementation may choose to use the latter, and thus avoid the possibility of unsoundness from arising out of a non-deterministic elaborator.

7 ANALYSIS

The surface language $\mathcal{X}$ allows us to easily manipulate existentials in a $\lambda$-calculus while delegating type consistency to an explicit core language $\mathcal{FX}$. The following theorems establish the soundness of this approach, via the elaboration transformation $\Rightarrow$, as well as the general expressivity and consistency of our bidirectional type system.

7.1 Soundness

If our surface language is to be type safe, we must know that any term accepted in the surface language corresponds to a well-typed term in the core language:

**Theorem 7.1 (Soundness).**

1. If $\Gamma \vdash e : \sigma \Rightarrow \epsilon \triangleleft \epsilon$, then $G \vdash e : s$, where $\Gamma \Rightarrow G$ and $\sigma \Rightarrow s$.
2. If $\Gamma \vdash e : \rho \Rightarrow \epsilon \triangleleft \epsilon$, then $G \vdash e : r$, where $\Gamma \Rightarrow G$ and $\rho \Rightarrow r$.
3. If $\Gamma \vdash e : \rho \Rightarrow \epsilon \triangleleft \epsilon$, then $G \vdash e : r$, where $\Gamma \Rightarrow G$ and $\rho \Rightarrow r$.

Furthermore, in order to eliminate the possibility of a trivial elaboration scheme, we would want the elaborated term to behave like the surface-language one. We capture this property in this theorem:

**Theorem 7.2 (Elaboration erasure).**

1. If $\Gamma \vdash e \triangleleft \sigma \Rightarrow \epsilon$, then $|e| = |e|$.
2. If $\Gamma \vdash e : \rho \Rightarrow \epsilon$, then $|e| = |e|$.
3. If $\Gamma \vdash e : \rho \Rightarrow \epsilon$, then $|e| = |e|$.

This theorem asserts that, if we remove all type annotations and applications, the $\mathcal{X}$ expression is the same as the $\mathcal{FX}$ one.

7.2 Conservativity

Not only do we want our $\mathcal{X}$ programs to be sound, but we also want $\mathcal{X}$ to be a comfortable language to program in. As our language is an extension of Hindley-Milner, we know that all the conveniences programmers are used to in that setting carry over here.

**Theorem 7.3 (Conservative extension of Hindley-Milner).** If $e$ has no type arguments or type annotations, and $\Gamma, e, \tau, \sigma$ contain no existentials, then:

1. $(\Gamma \vdash_{HM} e : \tau)$ implies $(\Gamma \vdash e : \tau)$
2. $(\Gamma \vdash_{HM} e : \sigma)$ implies $(\Gamma \vdash e : \sigma)$

where $\vdash_{HM}$ denotes typing in the Hindley-Milner type system, as described by Clément et al. [1986, Figure 3].

7.3 Stability

The following theorems denote stability properties [Bottu and Eisenberg 2021]. In other words, they ensure that small user-written transformations do not change drastically the static semantics.
Theorem 7.4 tells us that expanding out a well-typed let remains well typed. However, if we selectively expand a repeated let, a larger expression may become ill typed. Suppose we have \( f :: \text{Int} \rightarrow \exists a. (a, a \rightarrow \text{Int}) \) and write \( \text{snd} (f (\text{let } x = 5 \text{ in } x + x)) \) \( \text{fst} (f (\text{let } x = 5 \text{ in } x + x)) \). That expression is a well-typed Int. However, if we inline only one of the lets, to \( \text{snd} (f (5 + 5)) \) \( \text{fst} (f (\text{let } x = 5 \text{ in } x + x)) \), we now have an ill-typed expression. The problem is that our language uses a very fine-grained expression equality relation: just \( \alpha \)-equivalence. Accordingly, let \( x = 5 \text{ in } x + x \) and \( 5 + 5 \) are considered distinct, and when these expressions appear in types (via existential projections), the types are different.

The solution is straightforward, if not entirely lightweight: extend the expression equality relation. Doing so would require a more explicit treatment of equality in our type inference algorithm (in particular, rule APP of Figure 4 would need to invoke the equality relation), as well as additions to \( \text{FPX} \) to accommodate this new development. It is not clear whether the added expressiveness are worth the complexity cost, and so we kept our equivalence relationship simple for ease of presentation.

Aside 2. Selective let-inlining sometimes causes trouble

Theorem 7.4 (let-inlining). If \( x \) is free in \( e_2 \) then:

\[
\begin{align*}
\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 \Rightarrow \rho & \quad \text{implies} \quad \Gamma \vdash e_2[x / e_1] \Rightarrow \rho \\
\Gamma \upharpoonright \! \downarrow \text{let } x = e_1 \text{ in } e_2 \Leftarrow \sigma & \quad \text{implies} \quad \Gamma \upharpoonright \! \downarrow \! \downarrow e_2[x / e_1] \Leftarrow \sigma
\end{align*}
\]

Interestingly, the system we present here does not support a small generalization of the let-inlining property, as we explore in Aside 2.

This next theorem tells us that the order variables appear in an existential quantification does not affect usage sites:

Theorem 7.5 (Order of Quantification does not matter). Let \( \rho' \) (resp. \( \sigma' \)) be two types that differ from \( \rho \) (resp. \( \sigma \)) only by the ordering of quantified type variables in their (eventual) existential types. Then:

\[
\begin{align*}
(1) \quad (\Gamma \vdash e \Rightarrow \rho) & \quad \text{if and only if} \quad (\Gamma \vdash e \Rightarrow \rho') \\
(2) \quad (\Gamma \upharpoonright \! \downarrow \downarrow e \Leftarrow \sigma) & \quad \text{if and only if} \quad (\Gamma \upharpoonright \! \downarrow \downarrow e \Leftarrow \sigma')
\end{align*}
\]

Lastly, this theorem tells us that extra, redundant type annotations do not disrupt typability:

Theorem 7.6 (Synthesis implies checking). If \( \Gamma \vdash e \Rightarrow \rho \) then \( \Gamma \vdash e \Leftarrow \rho \).

8 INTEGRATING WITH TODAY’S GHC AND QUICK LOOK

We envision integrating our design into GHC, allowing Haskell programmers to use existential types in their programs. Accordingly, we must consider how our work fits with GHC’s latest type-inference algorithm, dubbed Quick Look [Serrano et al. 2020]. The structure behind our inference algorithm—with heads applied to lists of arguments instead of nested applications—is based directly on Quick Look, and it is straightforward to extend our work to be fully backward-compatible with that design. Indeed, our extension is essentially orthogonal to the innovations of impredicative type inference in the Quick Look algorithm.
\( \Gamma \vdash^\forall e \leftarrow \sigma \)  

**GenImpredicative**  
\[ \begin{array}{c} \bar{k} \text{ fresh} \\ \rho' = \rho[\bar{k} / \bar{b}] \\ \Gamma, \bar{a} \vdash e : \rho' \leadsto \Theta \\ \rho'' = \Theta \rho' \\ \text{dom}(\Theta) = \text{fiv}(\rho'') \\ \Gamma, \bar{a} \vdash e \leftarrow \theta \rho'' \end{array} \]  
\[ \Gamma \vdash^\forall e \leftarrow \forall \bar{a}. \exists \bar{b}, \rho \]

Fig. 10. Allowing impredicative instantiation in the \( \vdash^\forall \) judgment

It would take us too far afield from our primary goal—describing type inference for existential types—to explain the details of Quick Look here. We thus build on the text already written by Serrano et al. [2020]; readers uninterested in the details may safely skip the rest of this section.

Serrano et al. [2020] explains their algorithm progressively, by stating in their Figures 3 and 4 a baseline system. That baseline also effectively serves as our baseline here. Then, in their Figure 5, the authors add a few new premises to specific rules, along with judgments those premises refer to. Given this modular presentation, we can adopt the same changes: their rule \texttt{IArg} is our rule \texttt{IArg}, and their rule \texttt{App} is our rule \texttt{App}. The only wrinkle in merging these systems is that their presentation uses a notion of instantiation variable, which Serrano et al. write as \( \kappa \). An instantiation variable is allowed to unify with a polytype, in contrast to an ordinary unification variable, which must unify with a monotype. Given that impredicative instantiation is not a primary goal of our work, we choose not to use this approach in our main formal presentation, instead preferring the more conventional idiom of using guessed \( \tau \)-types. However, in order to integrate inferred existentials with Quick Look impredicativity, we must explicitly use instantiation variables in the rule below.

Since we have a more elaborate notion of polytype, one rule needs adjustment in our system: the rule implementing the \( \Gamma \vdash^\forall e \leftarrow \sigma \) judgment, rule \texttt{Gen}. That rule skolemizes (makes fresh constants out of) the variables universally quantified in \( \sigma \) and guesses \( \bar{\tau} \) to instantiate the existentially quantified variables. In order to allow these instantiations to be impredicative, we must modify the rule, as in Figure 10.

This rule follows broadly the pattern from rule \texttt{Gen}, but using instantiation variables \( \bar{k} \) instead of guessing \( \bar{\tau} \). The third premise invokes the Quick Look judgment \( \vdash_\iota \) [Serrano et al. 2020, Figure 5] to generate a substitution \( \Theta \). Such a substitution \( \Theta \) maps instantiation variables \( \kappa \) to polytypes \( \sigma \); by contrast, a substitution \( \theta \) includes only monotypes \( \tau \) in its codomain. The next two premises of rule \texttt{GenImpredicative} apply the \( \Theta \) substitution, and then use \( \theta \) to eliminate any remaining instantiation variables \( \kappa \): the \texttt{fiv}(\rho'') extracts all the free instantiation variables in \( \rho'' \). Note that the range of \( \theta \) appears unconstrained here; the types in its range are guessed, just like the \( \bar{\tau} \) in rule \texttt{Gen}.

With this one new rule—along with the changes evident in Figure 5 of Serrano et al.—our system supports impredicative type inference, and is a conservative extension of their algorithm.

9 DISCUSSION

We have described how our inference algorithm allows users to program with existentials while avoiding the need to thinking about packing and unpacking. Here, we review some subtleties that arise as our approach encounters more practical settings.
9.1 No Declarative (Non-syntax-directed) System with Existentials

When we first set out to understand type inference with existentials better, our goal was to develop a type system with existential types, unguided type inference (no additional annotation obligations for the programmer), and principal types. Our assumption was that if this is possible with universal quantification [Hindley 1969; Milner 1978], it should also be possible for existential quantification. Unfortunately, it seems such a design is out of reach.

To see why, consider $f \ b = \text{if } b \text{ then } (1, \lambda y \rightarrow y + 1) \text{ else } (\text{True}, \lambda z \rightarrow 1)$. We can see that $f$ can be assigned one of two different types:

(1) $\text{Bool} \rightarrow \exists a. (a, \text{Int} \rightarrow \text{Int})$

(2) $\text{Bool} \rightarrow \exists a. (a, a \rightarrow \text{Int})$

Neither of these types is more general than the other, and neither seems likely to be ruled out by straightforward syntactic restrictions (such as the Hindley-Milner type system’s requirement that all universal quantification be in prenex form).

One possible approach to inference for a definition like $f$ is to use an anti-unification [Pfenning 1991] algorithm to relate the types of $(1, \lambda y \rightarrow y + 1)$ and $(\text{True}, \lambda z \rightarrow 1)$: infer the former to have type $(\text{Int}, \text{Int} \rightarrow \text{Int})$ and the latter to have type $(\text{Bool}, a \rightarrow \text{Int})$ for some unknown type $a$. The goal then is to find some type $\tau$ such that $\tau$ can instantiate to either of these two types: this is anti-unification. The problem is, in this case, $a$: we get different results depending on whether $a$ becomes $\text{Int}$ or $\text{Bool}$.

We might imagine a way of choosing between the two hypothetical types for $f$, above, but any such restriction would break the desired symmetry and elegance of a declarative system that allows arbitrary generalization and specialization. Instead, we settle for the practical, predictable bidirectional algorithm presented in this paper, leaving the search for a more declarative approach as an open problem—one we think unlikely to have a satisfying solution.

9.2 Class Constraints on Existentials

The algorithm we present in this paper works with a typing context storing the types of bound variables. In full Haskell, however, we also have a set of constraint assumptions, and accepting some expressions requires proving certain constraints. A type system with these assumptions and obligations is often called a qualified type system [Jones 1992]. Our extension to support both universal and existential qualified types is in Figure 11.

This extension introduces type classes C and constraints Q. Constraints are applied type classes (like Show Int), and perhaps others; the details are immaterial. Instead, we refer to an abstract logical entailment relation $\vdash$, which relates assumptions and the constraints they entail. Universally quantified types $\sigma$ can now require proving a constraint: to use $e : Q \Rightarrow \sigma$, the constraint $Q$ must hold. Existentially quantified types $e$ can now provide the proof of a constraint: the expression $e : Q \land e$ contains evidence that $Q$ holds. Assumed constraints appear in contexts $\Gamma$.\textsuperscript{15}

The surprising feature here is that we have a new form of assumption, $[e : \epsilon]$. This assumption is allowed only when $\epsilon$ has the form $Q \land e'$; the assumed constraint is $Q$. However, by including the expression $e$ that proves $Q$ in the context, we remember how to compute $Q$ when it is required.

9.2.1 Static Semantics. Examining the typing rules, we see rule GenQualified assumes $Q_1$ as a given (following the usual treatment of givens in qualified type systems) and also assumes an arbitrary list of projections $[e : \epsilon]$. This arbitrary assumption is quite like how rule Gen assumes

\textsuperscript{15}Other presentations of qualified type systems frequently have a judgment that looks like $P \mid \Gamma \vdash e : \rho$, or similar, with a separate set of logical assumptions $P$. Because our assumptions may include expressions, we must mix the logical assumptions with variable assumptions right in the same context $\Gamma$.  

---

\textsuperscript{15}Other presentations of qualified type systems frequently have a judgment that looks like $P \mid \Gamma \vdash e : \rho$, or similar, with a separate set of logical assumptions $P$. Because our assumptions may include expressions, we must mix the logical assumptions with variable assumptions right in the same context $\Gamma$.
\[ C ::= \ldots \]
\[ Q ::= C\overline{\tau} | \ldots \]
\[ \sigma ::= e \mid \forall a.\sigma \mid Q \Rightarrow \sigma \]
\[ \epsilon ::= \rho \mid \exists b.\epsilon \mid Q \wedge \epsilon \]
\[ \Gamma ::= \emptyset \mid \Gamma, a \mid \Gamma, x:\sigma \mid \Gamma, Q \mid \Gamma, [e : \epsilon] \]
\[ \Gamma \models Q \]

**Fig. 11.** Type system extension to support existentially packed class constraints

The instantiation judgment \( \Gamma \vdash \) must also accommodate constraints. When, in rule IGIVEN, it comes across an expression whose type includes a packed assumption \( Q \wedge \epsilon \), it checks to make sure that assumption was included in \( \Gamma \). The design here requiring an arbitrary guess of assumptions, only to validate the guess later, is merely because our presentation is somewhat declarative. By contrast, an implementation would work by emitting constraints and solving them (that is, computing \( \models \)) later [Pottier and Rémy 2005]; when the constraint-generation pass encounters an expression of type \( Q \wedge \epsilon \), it simply emits the constraint as a given. Rule IWANTED is a straightforward encoding of the usual behavior of qualified types, where the usage of an expression of type \( Q \Rightarrow \sigma \) requires proving \( Q \).

9.2.2 Dynamic Semantics. An interesting new challenge with packed class constraints is that class constraints are not erasable. In practice, a function pretty of type Pretty \( a \Rightarrow a \rightarrow String \) takes two runtime arguments: a dictionary [Hall et al. 1996] containing implementations of the methods in Pretty, as well as the actual, visible argument of type \( a \). When this dictionary comes from an existential projection, the expression producing the existential will have to be evaluated.

For example, suppose we have \( mk :: Bool \rightarrow \exists a. Pretty a \wedge a \) and call pretty (mk True). Calling pretty requires passing the dictionary giving the the implementation of the function at the specific type pretty is instantiated at (\([mk \ True :: \exists a. Pretty a \wedge a]\), in this case). Getting this dictionary requires evaluating mk True. Naïvely, this means mk True would be evaluated twice. This makes some sense if we think of \( Q \wedge \epsilon \) as the type of pairs of a dictionary for \( Q \) and the inhabitant of \( \epsilon \): the naïve interpretation of pretty (mk True) thus is like calling pretty (fst (mk True)) (snd (mk True)).

We do not address how to do better here, as standard optimization techniques can apply to improve the potential repeated work. Once again, purity works to our advantage here, in that we can be assured that commoning up the calls to mk True does not introduce (or eliminate) effects.

9.3 Relevance and Existentials

One of the primary motivations for this work is to set the stage for an eventual connection between Liquid Haskell [Vazou et al. 2014] and the rest of Haskell’s type system. A Liquid Haskell refinement
type is exemplified by \{ v :: Int | v \geq 0 \}; any element of such a type is guaranteed to be non-negative. Yet what would it mean to have a function return such a type? To be concrete, let us imagine \( mk :: \text{Bool} \rightarrow \{ v :: \text{Int} | v \geq 0 \} \). This function would return a value \( v \) of type \( \text{Int} \), along with a proof that \( v \geq 0 \): this is a dependent pair, or an existential package. Thus, we can rephrase the type of \( mk \) to be \( \text{Bool} \rightarrow \exists (v :: \text{Int}). \) \( \text{Proof} \ (v \geq 0) \), where \( \text{Proof} \ q \) encodes a proof of the logical property \( q \).

However, our new form of existential is different than the others considered in this paper. Here, the relevant part is the first component, not the second. That is, we want to be able to project out \( v :: \text{Int} \) at runtime, discarding the compile-time proof that \( v \geq 0 \).

The core language presented in this paper cannot, without embellishment, support relevant first components of existentials. In other words, \([ e : e ]\) is always a compile-time type, never a runtime term. Nevertheless, existing approaches to deal with relevance will work in this new setting. Haskell’s \( \forall \) construct universally quantifies over an irrelevant type. Yet, work on dependent Haskell [Eisenberg 2016; Gundry 2013; Weirich et al. 2017] shows how we can make a similar, relevant construct. Similar approaches could work in a core language modeled on FX. Indeed, other dependently typed languages, such as Coq, Agda, and Idris support existential packages with relevant dependent components.

The big step our current work brings to this story is type inference. Whether relevant or not, we would still want existential packages to be packed and unpacked without explicit user direction, and we would still want type inference to have the properties of the algorithm presented in this paper. In effect, the choice of relevance of the dependent component is orthogonal to the concerns in this paper. We are thus confident that our approach would work in a setting with relevant types.

10 RELATED WORK

There is a long and rich body of literature informing our knowledge of existential types. We review some of the more prominent work here.

**History.** Existential types were present from the beginning in the design of polymorphic programming languages, present in Girard’s System F [Girard 1972] and independently discovered by Reynolds [1974], though in a less expressive form. Mitchell and Plotkin [1988] recognized the ability of existential types to model abstract datatypes and remarked on their connection with the \( \Sigma \)-types of Martin-Löf type theory [Martin-Löf 1975]. They proposed an elimination form, called \( \text{abstype} \), that is equivalent to the now standard \( \text{unpack} \).

Cardelli and Leroy [1990] compared Mitchell and Plotkin’s \( \text{unpack} \) based approach to various calculi with projection-based existentials. Their “calculus with a dot notation” includes the ability for the type language to project the type component from term variables of an existential type. At the end of the report (Section 4), they generalize to allow arbitrary expressions in projections. It is this language that is most similar to our core language. They also note a number of examples that are expressible only in this language.

**Integration with type inference.** Full type checking and type inference for domain-free System F with existential types is known to be undecidable [Nakazawa and Tatsuta 2009; Nakazawa et al. 2008]. As a result, several language designers have used explicit forms such as datatype declarations or type annotations to extend their languages with existential types.

The datatype-based version of existentials found in GHC was first suggested by Perry [1991] and implemented in Hope+. It was formalized by Läufer and Odersky [1994] and implemented in the Caml Light compiler for ML, along with the Haskell B compiler [Augustsson 1994].

The Utrecht Haskell Compiler (UHC) also supports a version of existential type [Dijkstra 2005], in a form that does not require the explicit connection to datatypes found in GHC. As in this work,
values of existential types can be opened in place, without the use of an \texttt{unpack} term. However, unlike here, UHC generates a fresh type variable for the abstracted type with each use of \texttt{open}. As a result, UHC does not need the form of dependent types that we propose, but also cannot express some of the examples allowed by our system (§3.3).

Leijen [2006] describes an extension of MLF [Le Botlan and Rémy 2003] with first-class existential types. Like this work, programmers never needed to add explicit \texttt{pack} or \texttt{unpack} expressions. However, because the type system was based on MLF, polymorphic types include instantiation constraints and the type-inference algorithm is very different from that used by GHC. In contrast, our work requires only a small extension of GHC’s most recent implementation of first-class polymorphism. Furthermore, Leijen does not describe a translation from his source language to an explicitly typed core language; a necessary implementation step for GHC.

Dunfield and Krishnaswami [2019] extend a bidirectional type system with indexes in existential types in order to support GADTs. As in this work, the introduction and elimination of existentials is implicit and determined by type annotations. Existentials are introduced via subsumption and eliminated via pattern matching. As a result, this type system has the same scoping limitations as one based on \texttt{unpack}.

In other contexts, if the domain of types that existentials are allowed to quantify over is restricted, more aggressive type inference is possible. For example, Tate et al. [2008] restrict existentials to hide only class types and develop a type-inference framework for a small object-oriented typed assembly language.

\textit{Module systems.} This paper also relates to work on ML-style module systems. We do not summarize that field here but mention some papers that are particularly inspirational or relevant.

MacQueen [1986] noted the deficiencies of Mitchell and Plotkin [1988] with respect to expressing modular structure. This work proposed the original form of the ML module system as a dependent type system based on strong \(\Sigma\)-types. As in our system, modules support projections of the abstracted type and values. However, unlike this work, the ML module language supports additional type system features: a phase separation between the compile-time and runtime parts of the language, a treatment of generativity which determines when module expressions should and should not define new types, etc, as described in Harper and Pierce [2005]. We do not intend to use this type system to express modular structure.

F-ing modules [Rossberg et al. 2014] present a formalization of ML modules using existential types and a translation of a module language into System \(F_{\omega}\) augmented with \texttt{pack} and \texttt{unpack}. Our approach is similar to theirs, in that we also use a translation of a surface language into our \(\text{FX}\). However, because the ML module system includes a phase separation, our concerns about strictness do not apply in that setting. As a result they can target the non-dependent language \(F_{\omega}\) and use \texttt{unpack} as their elimination form. Rossberg [2015] extends the source language to a more uniform design while still retaining the translation to a non-dependent core calculus.

Montagu and Rémy [2009] present an extension of System F to compute \texttt{open} existential types. They introduce the idea of decomposing the usual explicit \texttt{pack} and \texttt{unpack} constructs of System F, and we were inspired by those ideas to design the type system of our implicit surface language with opened existentials. Interestingly, for a long time, it was unknown whether full abstraction could be achieved with strong existentials. Crary [2017] plugged this hole, proving Reynold’s abstraction theorem for a module calculus based on strong \(\Sigma\)-types.

11 \textbf{CONCLUSION}

By leveraging strong existential types, we have presented a type-inference algorithm that can infer introduction and elimination sites for existential packages. Users can freely create and consume
existentials with no term-level annotations. The type annotation burden is small, and it dovetails with programmers’ current expectations around bidirectional type inference. The algorithm we present is designed to integrate well with GHC/Haskell’s state-of-the-art approach to type inference, the Quick Look algorithm [Serrano et al. 2020].

In order to prove our approach sound, we include an elaboration into a type-safe core language, inspired by Cardelli and Leroy [1990] and supporting the usual progress and preservation proofs. This core language is a small extension on System FC, the current core language implemented within GHC, and thus is suitable for implementation.

Beyond just soundness, we prove that inlining a let-binding preserves types, a non-trivial property in a type system with inferred existential types. We also prove that our type-inference algorithm is a conservative extension of a basic Hindley-Milner type system.

We believe and hope that our forthcoming implementation within GHC—in active development at the time of writing—will enable programmers to verify more aspects of their programs, even when that verification requires the use of existential types. We also hope that this new feature will provide a way forward to integrate the user-facing success of Liquid Haskell with GHC’s internal language and optimizer.

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A ELABORATION RULES

We first extend the FR grammar to include arguments:

\[ p ::= e \mid t \quad \text{argument} \]

(Elaboration for polymorphic expressions)

\[ \Gamma \vdash^\nu e \equiv \sigma \supset e \]

Elab-Gen

\[ \begin{align*}
\Gamma, a \vdash e & \equiv \rho[\tau / \overline{b}] \Rightarrow e \\
\imath & \Rightarrow t \\
\rho & \Rightarrow r \\
fv(\tau) & \subseteq \mathit{dom}(\Gamma, a)
\end{align*} \]

\[ \Gamma \vdash^\nu e \equiv \forall a.\exists \overline{b}.\rho \Rightarrow \Lambda a.\mathit{pack}[\tau], e \text{ as } \exists \overline{b}.r \]

(Elaboration for expressions)

\[ \Gamma \vdash e \Rightarrow \rho \Rightarrow e \quad \Gamma \vdash e \equiv \rho \Rightarrow e \]

Elab-App

\[ \Gamma \vdash h \Rightarrow \sigma \equiv h \]

\[ \Gamma \vdash_{\mathit{inst}} h : \sigma \Rightarrow \overline{\pi} \mathbin{\sim} \overline{\sigma}; \rho_r \equiv e_r \]

\[ \Gamma \vdash h\overline{\pi} \equiv \rho_r \Rightarrow e_r \]

Elab-cAbs

\[ \begin{align*}
\Gamma, x: \sigma & \vdash^\nu e \equiv \sigma_2 \Rightarrow e \\
fv(\sigma_1) & \subseteq \mathit{dom}(\Gamma) \\
\sigma_1 & \equiv s_1
\end{align*} \]

\[ \Gamma \vdash \lambda x.e \equiv \sigma_1 \rightarrow \sigma_2 \equiv \lambda x:s_1.e \]

(Elaboration for heads)

Elab-Var

\[ \begin{align*}
\Gamma \vdash x: \sigma & \in \Gamma \\
\Gamma \vdash h \Rightarrow \sigma \equiv x
\end{align*} \]

Elab-Ann

\[ \begin{align*}
\Gamma \vdash^\nu e & \equiv \sigma \Rightarrow e \\
fv(\sigma) & \subseteq \mathit{dom}(\Gamma)
\end{align*} \]

\[ \Gamma \vdash_{\mathit{inst}} (e :: \sigma) \Rightarrow \sigma \equiv e \]

(Elaboration for instantiation)

Elab-iAbs

\[ \begin{align*}
\Gamma \vdash h \Rightarrow \sigma & \equiv h \\
\Gamma \vdash e \Rightarrow \rho & \Rightarrow e
\end{align*} \]

Elab-App

\[ \begin{align*}
\Gamma \vdash h \Rightarrow \sigma & \equiv h \\
\Gamma \vdash_{\mathit{inst}} h : \sigma & \Rightarrow \overline{\pi} \mathbin{\sim} \overline{\sigma}; \rho_r \equiv e_r
\end{align*} \]

Elab-cAbs

\[ \begin{align*}
\Gamma, x: \sigma & \vdash^\nu e \equiv \sigma_2 \Rightarrow e \\
fv(\sigma_1) & \subseteq \mathit{dom}(\Gamma) \\
\sigma_1 & \equiv s_1
\end{align*} \]

\[ \Gamma \vdash \lambda x.e \equiv \sigma_1 \rightarrow \sigma_2 \equiv \lambda x:s_1.e \]

(Elaboration for heads)

Elab-Var

\[ \begin{align*}
\Gamma \vdash x: \sigma & \in \Gamma \\
\Gamma \vdash h \Rightarrow \sigma \equiv x
\end{align*} \]

Elab-Ann

\[ \begin{align*}
\Gamma \vdash^\nu e & \equiv \sigma \Rightarrow e \\
fv(\sigma) & \subseteq \mathit{dom}(\Gamma)
\end{align*} \]

\[ \Gamma \vdash_{\mathit{inst}} (e :: \sigma) \Rightarrow \sigma \equiv e \]

(Elaboration for instantiation)
Elab-ForAll
\[ \Gamma \vdash t : \pi \neq \sigma', \pi' \]
\[ \Gamma \vdash _\text{inst} e : \sigma[t/a] \Rightarrow e t ; \pi \sim \sigma ; \rho_e \Rightarrow e_r \]
\[ \Gamma \vdash _\text{inst} e : \forall \sigma.e \Rightarrow e ; \pi \sim \sigma ; \rho_r \Rightarrow e_r \]

Elab-
Elab-IExistCore
\[ \Gamma \vdash _\text{inst} e : \exists a.e \Rightarrow e ; \pi \sim \sigma ; \rho_e \Rightarrow e_r \]
Elab-IResult
\[ \Gamma \vdash _\text{inst} e : \rho_r \Rightarrow e_r ; \iota \sim \iota ; \rho_r \Rightarrow e_r \]

ElabC-Nil
\[ \emptyset \Rightarrow \emptyset \]
ElabC-TyVar
\[ \Gamma \Rightarrow \sigma \Rightarrow s \]
ElabC-Var
\[ \Gamma \Rightarrow \sigma \Rightarrow s \]

\[ \sigma \Rightarrow s \]
\[ \Gamma \Rightarrow G \]

In a small abuse of notation, we write (for example, in rule Elab-IABS) a list of types in a pack construct to denote nested packs. Formally, for \( e \) of type \( r[\overline{t} / \overline{a}] \), with \( \overline{t} = t_1 \ldots t_n \) and \( \overline{a} = a_1 \ldots a_n \), the construction is defined recursively by:

\[ \text{pack} t_1 \ldots t_n, e \text{ as } \exists a_1 \ldots a_n, r \Rightarrow \text{pack} t_1, (\text{pack} t_2 \ldots t_n, e \text{ as } \exists a_2 \ldots a_n, r[t_1 / a_1]) \text{ as } \exists a_1 \ldots a_n, r \]

Define erasure on \( \pi \) terms by the following equations:

\[ |n| = n \]
\[ |x| = x \]
\[ |e :: \sigma| = |e| \]
\[ |h \pi, e| = |h \pi| |e| \]
\[ |h \pi, \sigma| = |h \pi| \]
\[ |\lambda x.e| = \lambda x.|e| \]
\[ |\text{let } x = e_1 \text{ in } e_2| = \text{let } x = |e_1| \text{ in } |e_2| \]

Theorem A.1 (Elaboration erasure (Theorem 7.2)).

1. If \( \Gamma \vdash ^V e \Rightarrow \sigma \Rightarrow e \), then \( |e| = |e| \).
2. If \( \Gamma \vdash e \Rightarrow \rho \Rightarrow e \), then \( |e| = |e| \).
3. If \( \Gamma \vdash e \Rightarrow \rho \Rightarrow e \), then \( |e| = |e| \).
4. If \( \Gamma \vdash _h h \Rightarrow \sigma \Rightarrow h \), then \( |h| = |h| \).
5. If \( \Gamma \vdash _\text{inst} e : \sigma \Rightarrow e ; \pi \sim \sigma ; \rho_e \Rightarrow e_0 \) and \( |e| = |e| \), then \( |e\pi| = |e_0| \).

Proof. By straightforward induction on the elaboration judgments.
B PROOFS ABOUT OUR SURFACE LANGUAGE, X

Theorem B.1 (Soundness).

1. If $\Gamma \vdash^\forall e \equiv \sigma \Rightarrow e$, then $G \vdash e : s$, where $\Gamma \Rightarrow G$ and $\sigma \Rightarrow s$.
2. If $\Gamma \vdash e \Rightarrow \rho \Rightarrow e$, then $G \vdash e : r$, where $\Gamma \Rightarrow G$ and $\rho \Rightarrow r$.
3. If $\Gamma \vdash e \Rightarrow \rho \Rightarrow e$, then $G \vdash e : r$, where $\Gamma \Rightarrow G$ and $\rho \Rightarrow r$.
4. If $\Gamma \vdash h \Rightarrow \sigma \Rightarrow h$, then $G \vdash h : s$, where $\Gamma \Rightarrow G$ and $\sigma \Rightarrow s$.
5. If $\Gamma \vdash^\text{inst} h : \sigma \Rightarrow h ; \overline{\sigma} \Rightarrow e$, and $G \vdash h : s$, then $G \vdash e_r : r$, where $\Gamma \Rightarrow G$, $\sigma \Rightarrow s$ and $\rho_r \Rightarrow r_r$.

Proof. By (mutual) structural induction on the typing rule. The full set of rules can be found in Annex A.

Rule Elab-Gen From the premise: $\Gamma, \overline{\sigma} \vdash e \equiv \rho[\overline{\sigma} / \overline{\rho}] \Rightarrow e$, where $\overline{r} \Rightarrow t$ and $\rho \Rightarrow r$. By induction hypothesis, $G, \overline{\sigma} \vdash e : r[\overline{\sigma} / \overline{\rho}]$. By successive applications of rule CE-Pack we get $G, \overline{\sigma} \vdash \text{pack} \, \overline{\sigma} \, e \equiv \exists \overline{\sigma} \, r : \exists \overline{\sigma} \, x$. Then by successive applications of rule CE-Tabs we get the result: $G \vdash \overline{\sigma} \, \text{pack} \, \overline{\sigma} \, e \equiv \exists \overline{\sigma} \, r : \forall \overline{\sigma} \, \exists \overline{\sigma} \, x$.

Rule Elab-App Inference and synthesis are treated at the same time by mutual induction. By induction hypothesis, $G \vdash h : s$ where $\sigma \Rightarrow s$. Then by induction hypothesis (case (5)), we obtain $G \vdash e_r : r_r$.

Rule Elab-Tabs By induction hypothesis, $G, x : t \vdash e : r$. By applications of rule CE-Pack we obtain $G, x \vdash \text{pack} \, [r]_x e \equiv \exists \overline{\sigma} \, r' : \exists \overline{\sigma} \, r$ where $r' = r[\overline{\sigma} / [r]_x]$. We conclude by applying rule CE-Abs where the premise $x \notin \text{fv}(\exists \overline{\sigma} \, r')$ is verified by construction of $r'$ and definition of $[r]_x$.

Rule Elab-Cabs By induction hypothesis and rule CE-App.

Rule Elab-LetCore Inference and synthesis are treated at the same time. By induction hypothesis and rule rule CE-Let.

Rule Elab-Var Since $x : \sigma \in \Gamma$, we have $x : s \in G$ and we conclude by rule CE-Var.

Rule Elab-Ann By induction hypothesis.

Rule Elab-Infer By induction hypothesis.

We see the instantiation judgment for elaboration as a bottom-up computation initialized, in rule Elab-App, by a head $h$ such that $G \vdash h : s$. Hence we just prove that going "up" in the derivation tree maintains the invariant that the first core expression $e$ is well-typed (i.e. that $\Gamma \vdash^\text{inst} e : \sigma \Rightarrow e ; \overline{\sigma} \Rightarrow e_r$ implies $G \vdash e : s$ where $\sigma \Rightarrow s$).

Rule Elab-ityarg Assuming that $G \vdash e : \forall \ a.s$, by rule CE-TAPP: $G \vdash e' : s[s' / a]$.

Rule Elab-Iarg Assuming that $G \vdash e : s_1 \rightarrow s_2$ and $\Gamma \vdash^\forall e' \equiv \sigma_1 \Rightarrow e'$. By induction hypothesis, $G \vdash e' : s_1$ where $\sigma_1 \Rightarrow s_1$. By rule CE-App we obtain $G \vdash e : s_2$.

Rule Elab-Iall Assuming that $G \vdash e : \forall a.s$. By rule CE-TAPP, we obtain $G \vdash e : s[t / a]$.

Rule Elab-IXExistCore Assuming that $G \vdash e : \exists a.t$ where $e \Rightarrow t$. By rule CE-Open: $G \vdash \text{open} e : t[[e] / a]$.

Finally, at the top of the derivation tree, rule Elab-Result ensures that this invariant translates to the result of the computation, that is, to the second core expression $e_r$ and the result type $\rho_r$ such that $G \vdash e_r : r_r$ with $\rho_r \Rightarrow r_r$.

Theorem B.2 (Conservative extension of Clément et al. [1986]). If $e$ has no type arguments or type annotations, and $\Gamma, e, \tau, \sigma$ contain no existentials, then:

1. $(\Gamma \vdash^\text{HM} e : \tau)$ implies $(\Gamma \vdash e \Rightarrow \tau)$
2. $(\Gamma \vdash^\text{HM} e : \sigma)$ implies $(\Gamma \vdash^\forall e \equiv \sigma)$
where $\vdash_{HM}$ denotes typing in the Hindley-Milner type system, as described by Clément et al. [1986, Figure 3].

Proof. Proceed by induction on the length of the derivation for $\Gamma \vdash_{HM} e : \tau$ and case analysis on $e$.

$e = x$: The rule used is C_Var. From its premise we get $x: \forall \alpha.\tau' \in \Gamma$, with $\tau = \tau'[\tau/\alpha]$. In our type system, we can type $\Gamma \vdash_h x \Rightarrow \forall \alpha.\tau$ with H-Var. Then the instantiation judgment gives us $\Gamma \vdash_{inst} x : \forall \alpha.\tau' ; [] \Rightarrow [] ; \tau$ as the IAll rule will be used to instantiate $\forall \alpha.\tau$ with $\tau$. Finally we apply App to obtain $\Gamma \vdash x \Rightarrow \tau$.

$e = \lambda x.e'$: Since there are no existentials in $\tau = \tau_1 \rightarrow \tau_2$, hence in $\tau_2$, the iAbs rule is the same as the usual C_Abs rule, therefore we conclude by induction.

let $x = e_1$ in $e_2$: Without existentials, the Let rule is the same as applying the C_Gen and C_Let rules at the same time.

$e = h e_1 ... e_n$: The type of $h$ is $\tau_1 \rightarrow ... \rightarrow \tau_n \rightarrow \tau$. By applying the induction hypothesis on the successive premises obtained by inverting the C_App rules used to type $e$, we get $\Gamma \vdash e_i \Rightarrow \tau_i$ for all $i$, hence by Theorem 7.6: $\Gamma \vdash e_i \Leftarrow \tau_i$. The instantiation judgment, given as input $h : \tau_1 \rightarrow ... \rightarrow \tau_n \rightarrow \tau$ and the list of arguments $e_1 ... e_n$, outputs the list of types $\tau_1 ... \tau_n$ and the return type $\tau$. Hence we can apply App.

\[\Box\]

Theorem B.3 (Synthesis implies checking). If $\Gamma \vdash e \Rightarrow \rho$ then $\Gamma \vdash e \Leftarrow \rho$.

Proof. Proceed by induction on the typing judgment $\Gamma \vdash e \Rightarrow \rho$.

Rule iAbs: By inversion and applying the induction hypothesis, we get $\Gamma, x: \tau \vdash e \Leftarrow \rho$. Hence by rule Gen, $\Gamma, x: \tau \vdash e \Leftarrow \exists \alpha.\rho'$ and we conclude by rule cAbs.

Rule Let and rule App: Same rules for synthesis and checking.

\[\Box\]

Theorem B.4 (Order of Quantification does not matter). Let $\rho'$ (resp. $\sigma'$) be two types that differ from $\rho$ (resp. $\sigma$) only by the ordering of quantified type variables in their (eventual) existential types. Then:

1. $\Gamma \vdash e \Rightarrow \rho$ if and only if $\Gamma \vdash e \Rightarrow \rho'$
2. $\Gamma \vdash e \Leftarrow \sigma$ if and only if $\Gamma \vdash e \Leftarrow \sigma'$

Proof. In inference mode, the only rule that packs existentials is rule iAbs. This rule packs all the possible type variables at the same time, hence we see that their ordering does not matter. It is trivial therefore to choose one ordering or the other, to go from type $\rho$ to type $\rho'$.

In checking mode, rule Gen also does several packs at once, whose ordering does not matter. \[\Box\]

Lemma B.5. If $\alpha \notin \text{dom}(\Gamma)$

1. If $\Gamma \vdash^V e \Leftarrow \sigma$ then $\alpha \notin \text{fv}(e)$.
2. If $\Gamma \vdash^V e \Rightarrow \rho$ then $\alpha \notin \text{fv}(e)$.
3. If $\Gamma \vdash_h h \Rightarrow \sigma$ then $\alpha \notin \text{fv}(h)$.

Proof. By structural induction on the derivation.

Rule Gen: By inversion, $\Gamma, \alpha' \vdash e \Leftarrow \rho[\tau/\beta]$. By $\alpha$-equivalence, it is permissible to choose the $\alpha'$ fresh, such that $\alpha$ and $\alpha'$ do not intersect. Hence, we have $\alpha \notin \text{dom}(\Gamma, \alpha')$ and by induction hypothesis $\alpha \notin \text{fv}(e)$.

Lemma B.6. Assuming $\bar{a} \notin \text{dom}(\Gamma)$ and $\text{fv}(\tau) \subseteq \text{dom}(\Gamma)$.

(1) If $\Gamma \vdash^\tau e \equiv \sigma \Rightarrow e, \text{then } \Gamma \vdash^\tau e \iff \sigma[\tau/\bar{a}] \Rightarrow e[\bar{a}/\bar{a}], \text{where } \bar{a} \Rightarrow \bar{a}.$

(2) If $\Gamma \vdash^\tau h \Rightarrow \sigma \Rightarrow h, \text{then } \Gamma \vdash^\tau h \Rightarrow \sigma[\tau/\bar{a}] \Rightarrow h[\bar{a}/\bar{a}], \text{where } \bar{a} \Rightarrow \bar{a}.$

(3) If $\Gamma \vdash^\tau e \Leftarrow \rho \Rightarrow e, \text{then } \Gamma \vdash^\tau e \Leftarrow \rho[\tau/\bar{a}] \Rightarrow e[\bar{a}/\bar{a}], \text{where } \bar{a} \Rightarrow \bar{a}.$

(4) If $\Gamma \vdash^\tau h \Rightarrow \sigma[\tau/\bar{a}] \Rightarrow h[\bar{a}/\bar{a}] \text{ where } \bar{a} \Rightarrow \bar{a} \text{ and } \Gamma \vdash^\tau h : \sigma \Rightarrow h ; \bar{a} \Rightarrow \bar{a} ; \rho \Rightarrow e, \text{then } \Gamma \vdash^\tau h : \sigma[\tau/\bar{a}] \Rightarrow e[\bar{a}/\bar{a}] ; \bar{a} \Rightarrow \bar{a} ; \rho[\tau/\bar{a}] \Rightarrow e[\bar{a}/\bar{a}], \text{where } \bar{a} \Rightarrow \bar{a}.$

Proof. By structural induction on elaboration derivations. 

Rule Elab-Gen: Since $\bar{a} \notin \text{dom}(\Gamma, \bar{a}')$, by induction hypothesis $\Gamma, \bar{a}' + e \Rightarrow \rho[\tau/\bar{a}] \Rightarrow e[\bar{a}/\bar{a}]$ where $\bar{a} \Rightarrow \bar{a}$. By rule Elab-Gen $\Gamma \vdash^\tau e \equiv \forall \bar{a}' . \exists \bar{b} . \rho[\tau/\bar{a}] \Rightarrow \Lambda \bar{a} . \text{pack}(\bar{t}), e[\bar{a}/\bar{a}]$ as $\exists \bar{b} . \text{r}[\bar{t}/\bar{a}]$ where $\bar{t} \Rightarrow \bar{t}$. Since $\text{fv}(\tau) \subseteq \text{dom}(\Gamma, \bar{a}')$ and $\bar{a} \notin \text{dom}(\Gamma)$, $\Lambda \bar{a} . \text{pack}(\bar{t}), e[\bar{a}/\bar{a}]$ as $\exists \bar{b} . \text{r}[\bar{t}/\bar{a}] = (\Lambda \bar{a} . \text{pack}(\bar{t}), e[\bar{a}/\bar{a}]$ which concludes.

Rule Elab-App: By induction hypothesis and case (4) of the Lemma.

Rule Elab-1Abs: By induction hypothesis $\Gamma, x : \tau + e \Rightarrow \rho[\tau/\bar{a}] \Rightarrow e[\bar{a}/\bar{a}]$. We find that, since $\text{fv}(\bar{t}) \subseteq \text{dom}(\Gamma)$, $\rho[\tau/\bar{a}] \Rightarrow [\rho[\tau/\bar{a}]] = \rho[\bar{a}/\bar{a}]$ as $\bar{a} \Rightarrow \bar{a}$. So by rule Elab-1Abs, we obtain $\Gamma \vdash \lambda x . e \Rightarrow \tau \Rightarrow \exists \bar{a}, \rho[\tau/\bar{a}] \Rightarrow \lambda x . \text{pack}(\bar{t}), e[\bar{a}/\bar{a}]$ as $\exists \bar{a}' \Rightarrow \bar{a}'[\bar{a}/\bar{a}]$. 

Rule Elab-Cabs: By induction hypothesis. We also use $\text{fv}(\sigma) \subseteq \text{dom}(\Gamma)$ to prove $\lambda x : s_1 . e[\bar{a}/\bar{a}] = (\lambda x : s_1 . e)[\bar{a}/\bar{a}].$

Rule Elab-LetCore: After remarking that by construction of $\bar{a}' = \text{fv}(\rho_1) \setminus \text{dom}(\Gamma)$, $\forall \bar{a}' . \rho_1 = (\forall \bar{a}' . \rho_1)[\bar{a}/\bar{a}]$, we conclude by induction hypothesis.

Rule Elab-Var: Since $\bar{a} \notin \text{dom}(\Gamma)$, this means the $\bar{a}$ do not appear in $\sigma$ hence $\sigma[\bar{a}/\bar{a}] = \sigma$ and we are done.

Rule Elab-Ann: By induction hypothesis, and using the fact that $\text{fv}(\bar{t}) \subseteq \text{dom}(\Gamma)$.

Rule Elab-Infer: By induction hypothesis.

To prove case (4) of the Lemma, we go through the derivation tree for $\Gamma \vdash^\tau h : \sigma \Rightarrow h ; \bar{a} \Rightarrow \bar{a} ; \rho \Rightarrow e$ and transform it by applying the substitution $[\bar{t}/\bar{a}]$ at every intermediary step. We show that it does not change the result, since this substitution does not affect the application of the rules.

Rule Elab-Ityarg: Since $\text{fv}(\sigma') \subseteq \text{dom}(\Gamma)$ and $\bar{a} \notin \text{dom}(\Gamma)$, we conclude by noting that $\sigma[\bar{a}/\bar{a}] = \sigma[\sigma'/\bar{a}] [\bar{t}/\bar{a}]$.
**Rule Elab-IArg:** By case (1) of the Lemma, from $\Gamma \vdash^v e' \ll e \ll e'$ we obtain $\Gamma \vdash^v e' \ll 
abla[\tau / \bar{a}] \Rightarrow e' \nabla[\tau / \bar{a}]$. Hence we correctly have $\Gamma \vdash^v e' : \sigma_2[\bar{a}] \Rightarrow (e e') \nabla[\bar{a}] ; \bar{\pi} \Rightarrow 
abla[\bar{a}] ; \rho_1[\bar{a}] \Rightarrow e_r[\bar{a}]$.

**Rule Elab-IALL:** We just notice that, since $fv(\tau) \subseteq dom(\Gamma)$, $\sigma[\bar{a}] \Rightarrow \sigma[\tau / a] [\tau / a] = \sigma[\tau / a] [\bar{a}]$.

**Rule Elab-IEExistCore:** The rule applies with $\Gamma \vdash^v e : [e[\bar{a}] / \bar{a}] ; \rho_1[\bar{a}] \Rightarrow e[\bar{a}]$. We conclude by noting that $\epsilon([e[\bar{a}] / \bar{a}] ; \bar{a}] = \epsilon([e / a] ; [\bar{a}])$ and $e[\bar{a}] = (\epsilon[0] [\bar{a}])$.

**Rule Elab-IResult:** $\Gamma \vdash^v e : \rho_1[\bar{a}] \Rightarrow e_r[\bar{a}] ; [\bar{a}] \Rightarrow [\bar{a}] ; \rho_1[\bar{a}] \Rightarrow e_r[\bar{a}]$ is true.

□

**Lemma B.7 (Free variable substitution).** Given $a \notin dom(\Gamma)$:

1. If $\Gamma \vdash e \ll \sigma$, then $\Gamma \vdash e \ll \sigma[\bar{a}]$.
2. If $\Gamma \vdash h \Rightarrow \sigma$, then $\Gamma \vdash h \Rightarrow \sigma[\bar{a}]$.
3. If $\Gamma \vdash e \ll \rho$, then $\Gamma \vdash e \ll \rho[\bar{a}]$.
4. If $\Gamma \vdash h \Rightarrow \sigma$ and $\Gamma \vdash h : \sigma[\bar{a}] \Rightarrow \sigma[\bar{a}]$.

**Proof.** By corollary of Lemma B.6 □

**Lemma B.8 (Substitution).** Suppose $\Gamma_1 \vdash e_1 \Rightarrow \rho_1 \Rightarrow e_1$ and take $\bar{a} = fv(\rho_1) \backslash fv(\Gamma_1)$.

1. If $\Gamma_1, x : \forall \bar{a} \rho_1, \Gamma_2 \vdash e_2 \Rightarrow \rho_2$, then $\Gamma_1, \Gamma_2[\Lambda \bar{a} e_1 / x] \vdash e_2[\bar{a} / \bar{a}] \Rightarrow \rho_2[\Lambda \bar{a} e_1 / x]$.
2. If $\Gamma_1, x : \forall \bar{a} \rho_1, \Gamma_2 \vdash e_2 \ll \sigma$, then $\Gamma_1, \Gamma_2[\Lambda \bar{a} e_1 / x] \vdash e_2[\bar{a} / \bar{a}] \ll \sigma[\Lambda \bar{a} e_1 / x]$. (Since $\sigma[\bar{a}]$ does not appear in $\Gamma_1$, which is used to type $e_1$ with $\rho_1$, we have in fact $\rho_2[\Lambda \bar{a} e_1 / x] = \rho_1[\bar{a} / \bar{a}] [\bar{a} / \bar{a}]$. Thus, since $\Gamma_1 \vdash e_1 \ll \rho_1$ and $\bar{a} \notin dom(\Gamma_1)$, by Lemma B.7 we obtain $\Gamma_1 \vdash e_1 \ll \rho_2[\Lambda \bar{a} e_1 / x]$ and then we conclude by weakening.

3. If $\Gamma_1, x : \forall \bar{a} \rho_1, \Gamma_2 \vdash h \Rightarrow \sigma$, then $\Gamma_1, \Gamma_2[\Lambda \bar{a} e_1 / x] \vdash h[\bar{a} / \bar{a}] \Rightarrow \sigma[\Lambda \bar{a} e_1 / x]$. (Proof. (1,2,3,4) By induction on $e_2$.

**Proof.** 1. $e_2 = x$: Then $\Gamma_1, x : \forall \bar{a} \rho_1, \Gamma_2 \vdash h \ll \rho_2 \Rightarrow \rho_1[\bar{a} / \bar{a}]$. This means that $\rho_2[\Lambda \bar{a} e_1 / x] = \rho_1[\bar{a} / \bar{a}][\bar{a} / \bar{a}]$. Since $\sigma[\bar{a}]$ does not appear in $\Gamma_1$, which is used to type $e_1$ with $\rho_1$, we have in fact $\rho_2[\Lambda \bar{a} e_1 / x] = \rho_1[\bar{a} / \bar{a}] [\bar{a} / \bar{a}]$. Thus, since $\Gamma_1 \vdash e_1 \ll \rho_1$ and $\bar{a} \notin dom(\Gamma_1)$, by Lemma B.7 we obtain $\Gamma_1 \vdash e_1 \ll \rho_2[\Lambda \bar{a} e_1 / x]$ and then we conclude by weakening.

2. $e_2 = e : \sigma$: By inversion on rules APP and rule H-Ann, we get $\Gamma_1, x : \forall \bar{a} \rho_1 \vdash e \ll \sigma$. By induction hypothesis, $\Gamma_1 \vdash e[\bar{a} / \bar{a}] \ll \sigma[\Lambda \bar{a} e_1 / x]$. Then, since projections do not appear in type arguments, $\sigma[\Lambda \bar{a} e_1 / x] = \sigma$ and $\Gamma_1 \vdash e[\bar{a} / \bar{a}] : \sigma \Rightarrow \sigma$, and we conclude by applying rule APP.

3. $e_2 = \lambda y.e$: By inversion on rule IABS and induction hypothesis, $\Gamma_1 ; y : \tau[\Lambda \bar{a} e_1 / x] \vdash e[\bar{a} / \bar{a}] \Rightarrow \rho[\Lambda \bar{a} e_1 / x]$. Hence $\Gamma_1 \vdash \lambda y.e[\bar{a} / \bar{a}] \Rightarrow (\tau \Rightarrow \exists b.\rho')[\Lambda \bar{a} e_1 / x]$.

4. $e_2 = let y = e in e$: By the induction hypothesis.

5. $e_2 = h \bar{\pi}$ with non-empty $\bar{\pi}$: By the induction hypothesis.

□

**Theorem B.9 (Let-inlining).** If $x$ is free in $e_2$ then:

1. $(\Gamma \vdash let x = e_1 in e_2 \Rightarrow \rho)$ implies $(\Gamma \vdash e_2[\bar{a} / \bar{a}] \Rightarrow \rho)$
2. $(\Gamma \vdash let x = e_1 in e_2 \ll \sigma)$ implies $(\Gamma \vdash e_2[\bar{a} / \bar{a}] \ll \sigma)$
By inversion on the LetCore rule, we have
\[
\begin{align*}
\Gamma &\vdash e_1 \Rightarrow \rho_1 \Rightarrow e_1 \\
\Gamma, x \forall \alpha. \rho_1 \vdash e_2 \Rightarrow \rho' \\
\alpha & = fv(\rho_1) \setminus \text{dom}(\Gamma) \\
\rho & = \rho' [\Lambda \alpha.e_1 / x]
\end{align*}
\]
By Lemma B.8 we obtain \( \Gamma \vdash e_2[e_1/x] \Rightarrow \rho'[\Lambda \alpha.e_1 / x] \).

Let \( \sigma = \forall \alpha. \exists \beta. \rho \). By inversion on rule GEN, we have \( \Gamma, \alpha \vdash \text{let } x = e_1 \text{ in } e_2 \Leftarrow \rho[\tau / \beta] \). By inversion on rule LETCORE, we obtain:
\[
\begin{align*}
\Gamma, x \forall \alpha. \rho_1 &\vdash e_2 \Leftarrow \rho' [\Lambda \alpha.e_1 / x] \\
\Gamma &\vdash e_1 \Rightarrow \rho_1 \Rightarrow e_1 \\
\Gamma, x \forall \alpha. \rho_1 &\vdash e_2 \Leftarrow \rho' \\
\alpha & = fv(\rho_1) \setminus \text{dom}(\Gamma) \\
\rho & = \rho' [\Lambda \alpha.e_1 / x]
\end{align*}
\]
By Lemma B.8, we obtain \( \Gamma \vdash e_2[e_1/x] \Leftarrow \rho'[\Lambda \alpha.e_1 / x] \) i.e. \( \Gamma \vdash e_2[e_1/x] \Leftarrow \rho \). We conclude by rule GEN.

\[\square\]

C.1 Typing rules

(Core expression typing)

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CE-Var</td>
<td>( G \vdash e : t )</td>
</tr>
<tr>
<td>CE-Int</td>
<td>( G \vdash \text{Int} )</td>
</tr>
<tr>
<td>CE-Abs</td>
<td>( G \vdash \lambda x : t_1.e : t_2 )</td>
</tr>
<tr>
<td>CE-App</td>
<td>( G \vdash e_1 : t_1 \rightarrow t_2 )</td>
</tr>
<tr>
<td>CE-Pack</td>
<td>( G \vdash \text{pack } t, e \text{ as } \exists a.t_2 : \exists a.t_2 )</td>
</tr>
<tr>
<td>CE-TAbs</td>
<td>( G \vdash \exists a.e : \forall a.t )</td>
</tr>
<tr>
<td>CE-TApp</td>
<td>( G \vdash e : \forall a.t_1 )</td>
</tr>
<tr>
<td>CE-Open</td>
<td>( G \vdash \text{open } e : t[[\gamma] / a] )</td>
</tr>
<tr>
<td>CE-Let</td>
<td>( G \vdash \text{let } x = e_1 \text{ in } e_2 : t_2[e_1 / x] )</td>
</tr>
<tr>
<td>CE-Cast</td>
<td>( G \vdash e : t_1 \rightarrow t_2 )</td>
</tr>
</tbody>
</table>

(Core type well-formedness)

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
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<tbody>
<tr>
<td>CT-Var</td>
<td>( G \vdash a : \text{type} )</td>
</tr>
<tr>
<td>CT-Base</td>
<td>( G \vdash \text{Base } )</td>
</tr>
<tr>
<td>CT-ForAll</td>
<td>( G \vdash \forall a.t : \text{type} )</td>
</tr>
<tr>
<td>CT-Exists</td>
<td>( G \vdash \exists a.t : \text{type} )</td>
</tr>
<tr>
<td>CT-Proj</td>
<td>( G \vdash \text{proj } e \subseteq \text{dom}(G) )</td>
</tr>
</tbody>
</table>

G \vdash \text{type}
\[
\begin{align*}
\text{CG-Refl} & \quad \Gamma \vdash t : \text{type} \\
g & \vdash \gamma : \mathbf{t} \sim \mathbf{t}_2 \\
g \vdash \gamma : \mathbf{t}_1 \sim \mathbf{t}_2 \\
\text{CG-Trans} & \quad \Gamma \vdash \gamma : \mathbf{t}_1 \sim \mathbf{t}_2 \\
\text{CG-Base} & \quad \vdash \Gamma \vdash \mathbf{ok} \\
\text{CG-ForAll} & \quad \Gamma, a \vdash \gamma : \mathbf{t}_1 \sim \mathbf{t}_2 \\
\text{CG-Exists} & \quad \Gamma, a \vdash \gamma : \mathbf{t}_1 \sim \mathbf{t}_2 \\
\text{CG-Inst} & \quad \vdash \Gamma \vdash \eta : \mathbf{e}_1 \sim \mathbf{e}_2 \\
\text{CG-Proj} & \quad \vdash \mathbf{projpack} t, \mathbf{e} \sim \mathbf{t}_2 : \mathbf{t}_2 \\
\text{CG-Instr} & \quad \vdash \mathbf{pack} t, \mathbf{e} (\mathbf{as} \mathbf{t}_2) : \mathbf{t}_2 \\
\text{CG-Exists} & \quad \vdash \exists \mathbf{a} \gamma : (\exists \mathbf{a} \mathbf{t}_1) \sim (\exists \mathbf{a} \mathbf{t}_2) \\
\text{CG-Inst} & \quad \vdash \mathbf{nth}_n \gamma : \mathbf{t}_n \sim \mathbf{t}_n \\
\text{CH-Coh} & \quad \Gamma \vdash \gamma : \mathbf{t}_1 \sim \mathbf{t}_2 \\
\chi & \quad \Gamma \vdash \gamma : \mathbf{t}_1 \sim \mathbf{t}_2 \\
\text{CH-Step} & \quad \Gamma \vdash \mathbf{e} : \mathbf{t} \\
\text{C-Type} & \quad \vdash \mathbf{e} \rightarrow \mathbf{e}' \\
\text{C-App} & \quad \vdash \mathbf{app} \mathbf{v} \mathbf{e}_0 \\
\text{C-AppPull} & \quad \vdash \mathbf{sym} \mathbf{n} \mathbf{h} \gamma \mathbf{e} = \mathbf{yn} \mathbf{h} \\
\text{C-Tabs} & \quad \vdash \mathbf{a} \mathbf{e} \rightarrow \mathbf{a} \mathbf{e}' \\
\text{C-TabsPull} & \quad \vdash \mathbf{app} \mathbf{a} \mathbf{v} \gamma \rightarrow (\mathbf{a} \mathbf{v} \gamma) \rightarrow \mathbf{a} \mathbf{v} \gamma \\
\text{C-Tabs} & \quad \vdash \mathbf{a} \mathbf{e} \rightarrow \mathbf{a} \mathbf{e}' \\
\text{C-TabsPull} & \quad \vdash \mathbf{app} \mathbf{a} \mathbf{e} \rightarrow (\mathbf{a} \mathbf{e}) \rightarrow \mathbf{a} \mathbf{e}' \\
\text{C-Pack} & \quad \vdash \mathbf{pack} \mathbf{t} \mathbf{e} \rightarrow \mathbf{pack} \mathbf{t} \mathbf{e}' \rightarrow \mathbf{pack} \mathbf{t} \mathbf{e} \rightarrow \mathbf{pack} \mathbf{t} \mathbf{e}' \rightarrow \mathbf{pack} \mathbf{t} \mathbf{e} \rightarrow \mathbf{pack} \mathbf{t} \mathbf{e}' \rightarrow \mathbf{pack} \mathbf{t} \mathbf{e} \rightarrow \mathbf{pack} \mathbf{t} \mathbf{e}'
\end{align*}
\]

\[ G \vdash \text{open} (\text{pack} t, \nu \text{ as } t_2) \longrightarrow \nu \triangleright (t_2) \triangleright (\text{sym} (\text{projpack} t, \nu \text{ as } t_2)) \]

\[ G \vdash \text{open} (\text{pack} t, (\nu \triangleright \gamma) \text{ as } t_2) \longrightarrow (\nu \triangleright \gamma) \triangleright (t_2) \triangleright (\text{sym} (\text{projpack} t, (\nu \triangleright \gamma) \text{ as } t_2)) \]

\[ G \vdash e : t \quad G \vdash e \longrightarrow e' \]

\begin{align*}
G \vdash \text{open} e & \longrightarrow \text{open} e' \triangleright (t) \triangleright (\text{sym} [\text{step} e]) \\
G \vdash e & \longrightarrow e' \\
G \vdash \text{let} x = e_1 \text{ in } e_2 & \longrightarrow e_2[e_1 / x] \\
G \vdash (\nu \triangleright \gamma_1) \triangleright \gamma_2 & \longrightarrow \nu \triangleright (\gamma_1 \triangleright \gamma_2) \\
\end{align*}

\[ \text{CS-OpenPack} \]
\[ \text{CS-OpenPackCasted} \]
\[ \text{CS-OpenCong} \]
\[ \text{CS-OpenPull} \]
\[ \text{CS-Let} \]
\[ \text{CS-Cast} \]

\[ G \vdash e \longrightarrow e' \]

\[ G \vdash \text{let} x = e_1 \text{ in } e_2 \longrightarrow e_2[e_1 / x] \]

C.2 Structural properties

Lemma C.1 (Context regularity).
1. If \( G \vdash e : t \), then \( G \vdash \gamma. \)
2. If \( G \vdash t : \text{type} \), then \( G \vdash \gamma. \)
3. If \( G \vdash \gamma : t_1 \sim t_2 \), then \( G \vdash \gamma. \)
4. If \( G \vdash \eta : e_1 \sim e_2 \), then \( G \vdash \gamma. \)

Proof. By straightforward structural induction on the typing rule, inverting a rule in the context judgment in the cases of context extension. □

Lemma C.2 (Context prefix). If \( G \vdash G', \), then \( G \vdash G \)

Proof. Straightforward induction on the structure of \( G' \). □

Lemma C.3 (Weakening in types). If \( G \vdash t : \text{type and } G', \gamma. \) then \( G, \gamma \vdash t : \text{type} \).

Proof. By straightforward induction on \( G \vdash t : \text{type} \). In the case for rule \( \text{CT-Proj} \), we use the transitivity of \( \leq \). □

Lemma C.4 (Permutation in types). Suppose \( G' \) is a permutation of \( G \) and \( G', \gamma. \) If \( G \vdash t : \text{type, then } G' \vdash t : \text{type} \).

Proof. By straightforward induction on \( G \vdash t : \text{type} \). In the case for rule \( \text{CT-Proj} \), we use the fact that \( \leq \) ignores permutations. □

Lemma C.5 (Permutation in context prefixes). Suppose \( G' \) is a permutation of \( G \). If \( G \vdash G', \gamma. \) and \( G \vdash G', \gamma. \) then \( G, \gamma. \)


Lemma C.6 (Permutation in contexts (1)).
1. If \( G, x : t, a, G' \gamma. \) then \( G, a, x : t, G' \gamma. \)
2. If \( G, \, a, a', G' \gamma. \) then \( G, a, a', G' \gamma. \)

Proof.
(1) By Lemma C.2, we know \( \vdash G, x : t, a \text{ ok} \). Inversion tells us that \( G \vdash t : \text{ type} \). We then use rule C-TERM to get \( \vdash G, a, x : t \text{ ok} \). We are then done by Lemma C.5.

(2) By Lemma C.2, we know \( \vdash G, a', a \text{ ok} \). We are done by inversion, rule C-TYPE, and Lemma C.5.

\[ \square \]

**Lemma C.7 (Permutation in contexts).** If \( \vdash G_1, G_2, a, G_3 \text{ ok} \), then \( \vdash G_1, a, G_2, G_3 \text{ ok} \).

**Proof.** By induction on the structure of \( G_2 \), appealing to Lemma C.6.

\[ \square \]

**Lemma C.8 (Strengthening in contexts).** If \( \vdash G, x : t, G' \text{ ok} \) and \( G' \) contains only type variable bindings. Then \( \vdash G, G' \text{ ok} \).

**Proof.** Straightforward induction on the structure of \( G' \).

\[ \square \]

**Lemma C.9 (Strengthening in types).** Suppose \( G, x : t', G' \vdash t : \text{ type} \), \( x \notin \text{fv}(t) \), and \( G' \) contains only type variable bindings. Then \( G, G' \vdash t : \text{ type} \).

**Proof.** By induction on the structure of \( G, x : t', G' \vdash t : \text{ type} \).

**Rule CT-VAR:** By appeal to Lemma C.8 and rule CT-VAR.

**Rule CT-BASE:** By the induction hypothesis and Lemma C.8.

**Rule CT-FORALL:** By the induction hypothesis.

**Rule CT-EXISTS:** By the induction hypothesis.

**Rule CT-PROJ:** We use Lemma C.8 to show \( \vdash G, G' \text{ ok} \). We know \( t = \lfloor e \rfloor \), and that we further know that \( \text{fv}(e) \subseteq \text{dom}(G, x : t, G') \). However, we also have assumed that \( x \notin \text{fv}(e) \), and thus \( \text{fv}(e) \subseteq \text{dom}(G, G') \). We can finish with rule CT-PROJ.

\[ \square \]

**Lemma C.10 (Permutation in terms).** Suppose \( G' \) is a permutation of \( G \) and \( \vdash G' \text{ ok} \).

(1) If \( G \vdash e : t \), then \( G' \vdash e : t \).

(2) If \( G \vdash y : t_1 \sim t_2 \), then \( G' \vdash y : t_1 \sim t_2 \).

(3) If \( G \vdash \eta : e_1 \sim e_2 \), then \( G' \vdash \eta : e_1 \sim e_2 \).

(4) If \( G \vdash e \rightarrow e' \), then \( G' \vdash e \rightarrow e' \).

**Proof.** Straightforward mutual induction on the structure of the assumed typing judgment, using Lemma C.4 in cases that refer to the well-formedness of types.

\[ \square \]

**Lemma C.11 (Weakening in terms).** Suppose \( \vdash G, G' \text{ ok} \).

(1) If \( G \vdash e : t \), then \( G, G' \vdash e : t \).

(2) If \( G \vdash y : t_1 \sim t_2 \), then \( G, G' \vdash y : t_1 \sim t_2 \).

(3) If \( G \vdash \eta : e_1 \sim e_2 \), then \( G, G' \vdash \eta : e_1 \sim e_2 \).

(4) If \( G \vdash e \rightarrow e' \), then \( G, G' \vdash e \rightarrow e' \).

**Proof.** Straightforward mutual induction on the structure of the assumed judgment, allowing variable renaming in rules CE-ABS, CE-TABS, CE-LET, CG-FORALL, CG-EXISTS, and CS-TABS Cong and using Lemma C.10 in those cases. Cases using the type well-formedness judgment additionally need Lemma C.3.

\[ \square \]

**Lemma C.12 (Well-formed context types).** If \( \vdash G \text{ ok} \) and \( x : t \in G \) then \( \vdash t : \text{ type} \).

**Proof.** By structural induction on the structure of \( \vdash G \text{ ok} \).

**Rule C-NIL:** Not possible, by \( x : t \in G \).

**Rule C-TYPE:** By the induction hypothesis and Lemma C.3.
Rule C-TERM: If we have found the binding for \( x \), the result comes straight from Lemma C.3. Otherwise, we use the induction hypothesis and Lemma C.3.

Lemma C.13 (Expression scoping).

1. If \( G \vdash e : t \), then \( \text{fv}(e) \subseteq \text{dom}(G) \).
2. If \( G \vdash \gamma : t_1 \sim t_2 \), then \( \text{fv}(\gamma) \subseteq \text{dom}(G) \).
3. If \( G \vdash \eta : e_1 \sim e_2 \), then \( \text{fv}(\eta) \subseteq \text{dom}(G) \).

Proof. Straightforward mutual induction on \( G \vdash e : t \), \( G \vdash \gamma : t_1 \sim t_2 \), and \( G \vdash \eta : e_1 \sim e_2 \). We must use Lemma C.12 in the case for rule CE-ABS.

C.3 Preservation

Lemma C.14 (Type substitution in types).

1. If \( G_1, a, G_2 \vdash t_1 : \text{type} \) and \( G_1 \vdash t_2 : \text{type} \), then \( G_1, G_2[t_2 / a] \vdash t_1[t_2 / a] : \text{type} \).
2. If \( G \vdash G_1, a, G_2 \) \( \text{ok} \) and \( G_1 \vdash t_2 : \text{type} \), then \( G_1, G_2[t_2 / a] \) \( \text{ok} \).

Proof. By mutual induction on the structure of the typing judgments.

Rule CT-VAR: Here, we know \( t_1 = a' \), and inversion tells us \( \vdash G_1, a, G_2 \) \( \text{ok} \). The induction hypothesis tells us that \( \vdash G_1, G_2[t_2 / a] \) \( \text{ok} \). We now have three cases:
1. \( a' \in G_1 \): We must prove \( G_1, G_2[t_2 / a] \vdash \text{type} \). This comes straight from \( \vdash G_1, G_2[t_2 / a] \) \( \text{ok} \) and \( a' \in G_1 \), by rule CT-VAR.
2. \( a' = a \): We must prove \( G_1, G_2[t_2 / a] \vdash t_2 : \text{type} \). We are done by Lemma C.3.
3. \( a' \in G_2 \): We must prove \( G_1, G_2[t_2 / a] \vdash a' : \text{type} \). This comes straight from \( \vdash G_1, G_2[t_2 / a] \) \( \text{ok} \) and \( a' \in G_2 \), by rule CT-VAR. (Note that substitutions do not affect type variable bindings.)

Rule CT-BASE: By the induction hypothesis.

Rule CT-FORALL: By the induction hypothesis.

Rule CT-EXISTS: In this case, \( t_1 = \exists a'.t_0 \). Inversion tells us \( \vdash G_1, a, G_2, a' \vdash t_0 : \text{type} \). We now use the induction hypothesis to get \( G_1, G_2[t_2 / a], a' \vdash t_0[t_2 / a] : \text{type} \) and finish with rule CT-EXISTS to get \( G_1, G_2[t_2 / a] \vdash \exists a'.t_0[t_2 / a] : \text{type} \) as desired.

Rule CT-PROJ: We know \( t_1 = [e] \), and inversion tells us that \( \vdash G_1, a, G_2 \) \( \text{ok} \) and \( \text{fv}(e) \subseteq \text{dom}(G_1, a, G_2) \). We must prove \( G_1, G_2[t_2 / a] \vdash [e[t_2 / a]] : \text{type} \). The induction hypothesis tells us that \( \vdash G_1, G_2[t_2 / a] \) \( \text{ok} \), so (using rule CT-PROJ) we must prove only that \( \text{fv}(e[t_2 / a]) \subseteq \text{dom}(G_1, G_2[t_2 / a]) \). This must be true, because \( a \) cannot be free in \( e[t_2 / a] \) and \( \text{dom}(G_2[t_2 / a]) = \text{dom}(G_2) \).

Rule C-NIL: Impossible.

Rule C-TYPE: We have two cases, depending on whether \( G_2 \) is empty. If \( G_2 \) is empty, our result is immediate. Otherwise, it comes from the induction hypothesis.

Rule C-TERM: By the induction hypothesis.

Lemma C.15 (Type substitution).

1. If \( G_1, x : t_2, G_2 \vdash t_1 : \text{type} \) and \( G_1 \vdash e_2 : t_2 \), then \( G_1, G_2[e_2 / x] \vdash t_1[e_2 / x] : \text{type} \).
2. If \( G \vdash G_1, x : t_2, G_2 \text{ok} \) and \( G_1 \vdash e_2 : t_2 \), then \( G_1, G_2[e_2 / x] \) \( \text{ok} \).

Proof. By mutual induction on the typing judgments.

Rule CT-VAR: We know that \( t_1 = a \), and inversion of rule CT-VAR gives us \( \vdash G_1, x : t_2, G_2 \) \( \text{ok} \) and \( a \in G_1, x : t_2, G_2 \). We must prove \( G_1, G_2[e_2 / x] \vdash a : \text{type} \). The induction hypothesis
gives us that \( \vdash G_1, G_2[e_2 / x] \text{ ok} \). And, noting that substitutions do not affect type variable bindings, we must have \( a \in G_1, G_2[e_2 / x] \). Thus we are done by rule CT-VAR.

**Rule CT-BASE:** By the induction hypothesis.

**Rule CT-FORALL:** By the induction hypothesis.

**Rule CT-EXISTS:** By the induction hypothesis.

**Rule CT-PROJ:** We know that \( t_1 = [e] \); we must prove \( G_1, G_2[e_2 / x] \vdash [e][e_2 / x] : \text{ type} \).

<table>
<thead>
<tr>
<th>We know</th>
<th>How</th>
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<tbody>
<tr>
<td>( \vdash G_1, x : t_2, G_2 \text{ ok} )</td>
<td>inversion of rule CT-PROJ</td>
</tr>
<tr>
<td>( \text{fv}(e) \subseteq \text{dom}(G_1, x : t_2, G_2) )</td>
<td>inversion of rule CT-PROJ</td>
</tr>
<tr>
<td>( \vdash G_1, G_2[e_2 / x] \text{ ok} )</td>
<td>induction hypothesis</td>
</tr>
<tr>
<td>( \text{fv}(e[e_2 / x]) \subseteq \text{fv}(e) \cup \text{fv}(e_2) \setminus {x} )</td>
<td>def’n of substitution</td>
</tr>
<tr>
<td>( \text{fv}(e[e_2 / x]) \subseteq \text{dom}(G_1, G_2[e_2 / x]) )</td>
<td>rules of ( \subseteq )</td>
</tr>
<tr>
<td>( G_1, G_2[e_2 / x] \vdash [e][e_2 / x] : \text{ type} )</td>
<td>rule CT-PROJ</td>
</tr>
</tbody>
</table>

**Rule C-NIL:** Impossible, as the starting context is not empty (it has a binding for \( x \)).

**Rule C-TYPE:** By the induction hypothesis, noting that the substitution in contexts will not affect a type variable binding. (Type variables \( a \) and term variables \( x \) are distinct.)

**Rule C-TERM:** We have two cases: either \( G_2 \) is empty or not. If it is empty, then we are done by Lemma C.1. If it is not empty, then we know that the substitution does not affect the name of the last variable in the context, and we are done by the (first) induction hypothesis.

**Lemma C.16 (Substitution in values).** If \( \nu \) is a value, then \( \nu[e / x] \) is also a value.

**Proof.** Straightforward induction on the definition of values.

**Lemma C.17 (Substitution).** Suppose \( G_1 \vdash e_2 : t_2 \).

1. If \( G_1, x : t_2, G_2 \vdash e_1 : t_1 \), then \( G_1, G_2[e_2 / x] \vdash e_1[e_2 / x] : t_1[e_2 / x] \).
2. If \( G_1, x : t_2, G_2 \vdash \gamma : t_0 \sim t_1 \), then \( G_1, G_2[e_2 / x] \vdash \gamma[e_2 / x] : t_0[e_2 / x] \sim t_1[e_2 / x] \).
3. If \( G_1, x : t_2, G_2 \vdash \eta \in e_0 \sim e_1 \), then \( G_1, G_2[e_2 / x] \vdash \eta[e_2 / x] : e_0[e_2 / x] \sim e_1[e_2 / x] \).
4. If \( G_1, x : t_2, G_2 \vdash e_1 \rightarrow e'_1 \), then \( G_1, G_2[e_2 / x] \vdash e_1[e_2 / x] \rightarrow e'_1[e_2 / x] \).

**Proof.** By mutual induction on the structure of \( G_1, x : t_2, G_2 \vdash e_1 : t_1 \), \( G_1, x : t_2, G_2 \vdash \gamma : t_0 \sim t_1 \), and \( G_1, x : t_2, G_2 \vdash \eta : e_0 \sim e_1 \).

**Rule CE-VAR:** Here, \( e_1 = x' \) for some \( x' \). We have three cases:

1. \( x' : t_1 \in G_1 \): By Lemma C.1, we know that \( x \not\in \text{dom}(G_1) \). Thus, \( x' \not\in x' \). Thus, \( e_1[e_2 / x] = e_1 = x' \). We now must show that \( t_1 \) does not mention \( x \). This comes from the fact that \( t_1 \) is well-formed within \( G_1 \) (Lemma C.12) and thus that \( \text{fv}(t_1) \subseteq \text{dom}(G_1) \), excluding \( x \). We have now established that \( t_1[e_2 / x] = t_1 \). Our final goal is thus \( G_1, G_2[e_2 / x] \vdash x' : t_1 \); we know \( x' : t_1 \in G_1 \). To use rule CE-VAR, we must only show \( \vdash G_1, G_2[e_2 / x] \text{ ok} \). This comes straight from Lemma C.15, and we are done with this case.
2. \( x' = x \): Using Lemma C.15 to get \( \vdash G_1, G_2[e_2 / x] \text{ ok} \), we are done by Lemma C.11.
3. \( x' : t_1 \in G_2 \): We know \( x' \not\in x' \) by the well-formedness of the context. We must show \( G_1, G_2[e_2 / x] \vdash x' : t_1[e_2 / x] \). Since \( x' : t_1 \in G_2 \), then it must be that \( x' : t_1[e_2 / x] \in G_2[e_2 / x] \). We are thus done by rule CE-VAR and Lemma C.15.

**Rule CE-INT:** Direct from Lemma C.15, noting that the substitutions in the subject and object have no effect.

**Rule CE-Abs:** Here, \( e_1 = \lambda x' : t_4, e_3 \) for some \( x', t_3, \) and \( e_3 \). We also have \( t_1 = t_3 \rightarrow t_4 \) for some \( t_4 \) such that \( G_1, x : t_1, G_2, x' : t_3 \vdash e_3 : t_4 \). The induction hypothesis tells us that \( G_1, G_2[e_2 / x], x' : t_3[e_2 / x] \vdash e_3[e_2 / x] : t_4[e_2 / x] \). This is exactly what we need to use...
rule CE-ABS, and we are thus done (noting that it must be that \(fv(e_2)\) does not include \(x'\), as \(x'\) is locally bound).

**Rule CE-APP:** By the induction hypothesis.

**Rule CE-TABS:** By the induction hypothesis.

**Rule CE-TAPP:** By the induction hypothesis and Lemma C.15.

**Rule CE-PACK:** Here, \(e_1 = \text{pack } t, e\) as \(\exists \ a.t'\), where \(t_1 = \exists \ a.t'\). We must show \(G_1, G_2[e_2 / x] \vdash \text{pack } t[e_2 / x], e[e_2 / x] \alpha \exists \ a.t'[e_2 / x] : \exists \ a.t'[e_2 / x].\) Lemma C.15 gives us the first two premises of rule CE-PACK. We must show \(G_1, G_2[e_2 / x] \vdash e[e_2 / x] : t'[e_2 / x][t[e_2 / x] / a].\) By the algebra of substitutions, the object of this judgment equals \(t'[e_2 / x] [t[e_2 / x] / a].\) By inversion on our original assumption, we know \(G_1, x : t_2, G_2 \vdash e : t'[t / a].\) We are thus done by the induction hypothesis.

**Rule CE-OPEN:** Here, \(e_1 = \text{open } e\), where \(G_1, x : t_2, G_2 \vdash e : \exists \ a.t\) and \(t_1 = t[\ell e / a].\) We must show \(G_1, G_2[e_2 / x] \vdash \text{open } e[e_2 / x] : t[\ell e / a][e_2 / x] \alpha \exists \ a.t[e_2 / x].\) The object of this judgment equals \(t[e_2 / x][\ell e / a].\) To use rule CE-OPEN, we must show \(G_1, G_2[e_2 / x] \vdash e[e_2 / x] : \exists \ a.t[e_2 / x].\) This comes directly from the induction hypothesis, and so we are done with this case.

**Rule CE-LET:** Similar to the case for rule CE-ABS.

**Rule CE-CAST:** By the induction hypothesis.

**Rule CG-REFL:** By Lemma C.15.

**Rule CG-SYM:** By the induction hypothesis.

**Rule CG-TRANS:** By the induction hypothesis.

**Rule CG-BASE:** By the induction hypothesis and Lemma C.15.

**Rule CG-FORALL:** By the induction hypothesis.

**Rule CG-EXISTS:** By the induction hypothesis.

**Rule CG-PROJ:** By the induction hypothesis.

**Rule CG-PROJPACK:** By the induction hypothesis.

**Rule CG-INSTFORALL:** By the induction hypothesis, noting that the substitutions commute, as their domains are distinct.

**Rule CG-INSTEXISTS:** By the induction hypothesis, noting that the substitutions commute, as their domains are distinct.

**Rule CG-NTH:** By the induction hypothesis.

**Rule CH-COHERENCE:** By the induction hypothesis.

**Rule CH-STEP:** By the induction hypothesis.

**Rule CS-BETA:** We know \(e_1 = (\lambda x_0.t e_3) e_4\) and \(e_1' = e_5[e_4 / x_0].\) We must show \(G_1, G_2[e_2 / x] \vdash (\lambda x_0.t[e_2 / x], e_3[e_2 / x]) e_4[e_2 / x] \rightarrow e_3[e_4 / x_0][e_2 / x].\) Rule CS-BETA tells us \(G_1, G_2[e_2 / x] \vdash (\lambda x_0.t[e_2 / x], e_3[e_2 / x]) e_4[e_2 / x] \rightarrow e_3[e_2 / x][e_4[e_2 / x] / x_0].\) A little algebra on substitutions (and the fact that \(x \neq x_0\), renaming if necessary) shows that these judgments are the same.

**Rule CS-APPCONG:** By the induction hypothesis.

**Rule CS-APPULL:** By the induction hypothesis.

**Rule CS-TABS:** By the induction hypothesis.

**Rule CS-TAPPCONG:** By Lemma C.16.

**Rule CS-TBETA:** Similar to the case for rule CS-BETA, with an appeal to Lemma C.16.

**Rule CS-TAPPPULL:** By the induction hypothesis.

**Rule CS-PACK:** By the induction hypothesis.

**Rule CS-OPENPACK:** By Lemma C.16.

**Rule CS-OPENCASTED:** By Lemma C.16.
Lemma C.18 (Type substitution in terms). Suppose $G_1, a, G_2 ⊢ e_1 : t_1$, then $G_1, G_2[t_2 / a] ⊢ e_1[t_2 / a] : t_1[t_2 / a]$.

Proof. By mutual induction on the structure of $G_1, a, G_2 ⊢ e_1 : t_1$, $G_1, a, G_2 ⊢ e_0 : e_1, then G_1, G_2[t_2 / a] ⊢ e_0[t_2 / a] : e_1[t_2 / a]$.

Rule CS-Var: Here, $e_1 = x$ for some $x$. We have two cases:

1. $x : t_1 ∈ G_1$: Similar to the reasoning in this case in the proof of Lemma C.17, but invoking Lemma C.14.
2. $x : t_1 ∈ G_2$: Similar to the reasoning in this case in the proof of Lemma C.17, but invoking Lemma C.14.


Rule CE-Tab: By the induction hypothesis.

Rule CE-Tapp: By the induction hypothesis and Lemma C.14.

Rule CE-Pack: Similar to this case in the proof of Lemma C.17, using Lemma C.14.

Rule CE-Open: Similar to this case in the proof of Lemma C.17.

Rule CE-Let: Similar to this case in the proof of Lemma C.17.

Rule CE-Cast: By the induction hypothesis.

Rule CG-Ref1: By Lemma C.14.

Rule CG-Sym: By the induction hypothesis.

Rule CG-Trans: By the induction hypothesis.

Rule CG-Base: By the induction hypothesis and Lemma C.14.

Rule CG-ForAll: By the induction hypothesis.

Rule CG-Exists: By the induction hypothesis.

Rule CG-Proj: By the induction hypothesis.

Rule CG-ProjPack: By the induction hypothesis.

Rule CG-InstForAll: By the induction hypothesis, noting that the substitutions commute as their domains are distinct (renaming the local bound variable, if necessary).

Rule CG-InstExists: By the induction hypothesis, noting that the substitutions commute as their domains are distinct (renaming the local bound variable, if necessary).

Rule CG-Nth: By the induction hypothesis.

Rule CH-Coherence: By the induction hypothesis.

Rule CH-Step: By the induction hypothesis.

Cases for $G_1, a, G_2 ⊢ e → e'$: Similar to these cases in the proof of Lemma C.17.

Lemma C.19 (Object regularity).

1. If $G ⊢ e : t$, then $G ⊢ t : type$. 

(2) If \( G \vdash \gamma : t_1 \sim t_2 \), then \( G \vdash t_1 : \text{type} \) and \( G \vdash t_2 : \text{type} \).

(3) If \( G \vdash \eta : e_1 \sim e_2 \), then there exist \( t_1 \) and \( t_2 \) such that \( G \vdash e_1 : t_1 \) and \( G \vdash e_2 : t_2 \).

**Proof.** By mutual structural induction on the typing judgments. Note that we know \( \vdash G \text{ok} \) by Lemma C.1.

**Rule CE-VAR:** By Lemma C.12.

**Rule CE-INT:** Trivial, by rule CT-BASE.

**Rule CE-Abs:** Here, we know \( t = t_1 \rightarrow t_2 \). We know \( \vdash G, x : t_1 \text{ok} \) by Lemma C.1. Thus, by Lemma C.12, we have \( G \vdash t_1 : \text{type} \). The induction hypothesis gives us \( G, x : t_1 \vdash t_2 : \text{type} \), but we also know that \( x \notin fv(t_2) \). We can use Lemma C.9 to get \( G \vdash t_2 : \text{type} \), and we are done by rule CT-BASE.

**Rule CE-APP:** By the induction hypothesis, inverting rule CT-BASE.

**Rule CE-TABS:** By the induction hypothesis and rule CT-FORALL.

**Rule CE-OPEN:** Here, we know \( e = e_1 \cdot t_2 \), where \( t = t_1[t_2 / a] \) and \( G \vdash e_1 : \forall a.t_1 \) and \( G \vdash t_2 : \text{type} \). We must show \( G \vdash t_1[t_2 / a] : \text{type} \); we are thus done by Lemma C.14.

**Rule CE-Pack:** By inversion.

**Rule CE-OPEN:** We know \( e = \text{open} e_0 \), and (by inversion) \( G \vdash e_0 : \exists a.t_0 \). We must prove \( G \vdash t_0[[e_0] / a] : \text{type} \). The induction hypothesis tells us that \( G \vdash \exists a.t_0 : \text{type} \). Inversion by rule CT-EXISTS then tells us \( G, a \vdash t_0 : \text{type} \). To use Lemma C.14, we must now show \( G \vdash [e_0] : \text{type} \). To use rule CT-PROJ, we must now show the following:

\[
\vdash G \text{ok}: \text{This is from Lemma C.1}.
\]

\( \text{fv}(e_0) \subseteq \text{dom}(G) \): This is from Lemma C.13.

Rule CT-PROJ gives us \( G \vdash [e_0] : \text{type} \) and then Lemma C.14 gives us \( G \vdash t_0[[e_0] / a] : \text{type} \) as desired.

**Rule CE-LET:** By the induction hypothesis and Lemma C.15.

**Rule CE-CAST:** By the induction hypothesis.

**Rule CG-REFL:** By inversion.

**Rule CG-SYM:** By the induction hypothesis.

**Rule CG-TRANS:** By the induction hypothesis.

**Rule CG-BASE:** By the induction hypothesis and rule CT-BASE.

**Rule CG-FORALL:** By the induction hypothesis and rule CT-FORALL.

**Rule CG-EXISTS:** By the induction hypothesis and rule CT-EXISTS.

**Rule CG-INST:** By the induction hypothesis, Lemma C.13, and rule CT-PROJ.

**Rule CG-PROJ:** Here, \( \gamma = \text{projpack} t_3, e \text{ as } t_4 \), and we must show \( G \vdash \{ \text{pack} t_3, e \text{ as } t_4 \} : \text{type} \) and \( G \vdash t_3 : \text{type} \). Inversion on the typing judgment gives us \( G \vdash \text{pack} t_3, e \text{ as } t_4 : t_4 \). This can be so only by rule CE-Pack. We can thus invert again to get \( G \vdash t_3 : \text{type} \). We use Lemma C.13 and we are done by rule CT-PROJ.

**Rule CG-INSTFORALL:** In this case, we know \( \gamma = \gamma_1 @ \gamma_2 \), with inversion giving us \( G \vdash \gamma_1 : \forall a.t_3 \sim (\forall a.t_4) \) and \( G \vdash \gamma_2 : t_5 \sim t_6 \). We must show \( G \vdash t_3[t_5 / a] : \text{type} \) and \( G \vdash t_4[t_6 / a] : \text{type} \). Let’s focus on the first of these.

We know

\[
\begin{align*}
G & \vdash \forall a.t_3 : \text{type} \\
G, a & \vdash t_3 : \text{type} \\
G & \vdash t_5 : \text{type} \\
G, a & \vdash t_3[t_5 / a] : \text{type} \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>How</th>
</tr>
</thead>
<tbody>
<tr>
<td>induction hypothesis</td>
</tr>
<tr>
<td>inversion of rule CT-FORALL</td>
</tr>
<tr>
<td>induction hypothesis</td>
</tr>
<tr>
<td>Lemma C.14</td>
</tr>
</tbody>
</table>

The derivation for \( G \vdash t_4[t_6 / a] : \text{type} \) is similar.
An Existential Crisis Resolved

Theorem C.20 (Preservation). If $G \vdash e : t$ and $G \vdash e \rightarrow e'$, then $G \vdash e' : t$.

Proof. By induction on the structure of $G \vdash e \rightarrow e'$.

Rule CS-BETA: We have $e = (\lambda x : t_1. e_1) e_2$ and $e' = e_1[e_2/x]$, and we know $G \vdash \lambda x : t_1. e_1 : t_1 \rightarrow t_2$ (with our original type $t$ equalling $t_2$) and $G \vdash e_2 : t_1$. The former must be by rule CE-ABS, and we can thus conclude $G, x : t_1 \vdash e_1 : t_2$ and $x \notin fv(t_2)$. Lemma C.17 tells us $G \vdash e_1[e_2/x] : t_2[e_2/x]$. But since $x \notin fv(t_2)$, this reduces to $G \vdash e_1[e_2/x] : t_2$, and we are done with this case.

Rule CS-APPCONG: By the induction hypothesis.

Rule CS-TABS Cong: By the induction hypothesis.
**Rule CS-TAbsPull:** In this case, we know $e = \Lambda a.(v \triangleright \gamma)$. We must prove $G \vdash (\Lambda a.v) \triangleright \forall a.\gamma : t$.

<table>
<thead>
<tr>
<th>We know</th>
<th>How</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G \vdash \Lambda a.(v \triangleright \gamma) : t$</td>
<td>assumption</td>
</tr>
<tr>
<td>$G, a \vdash v \triangleright \gamma : t_1$</td>
<td>inversion of rule CE-TAbs</td>
</tr>
<tr>
<td>$t = \forall a.t_1$</td>
<td>inversion of rule CE-TAbs</td>
</tr>
<tr>
<td>$G, a \vdash v : t_2$</td>
<td>inversion of rule CE-Cast</td>
</tr>
<tr>
<td>$G, a \vdash \gamma : t_2 \sim t_1$</td>
<td>inversion of rule CE-Cast</td>
</tr>
<tr>
<td>$G \vdash \forall a.\gamma : (\forall a.t_2) \sim (\forall a.t_1)$</td>
<td>rule CG-ForAll</td>
</tr>
<tr>
<td>$G \vdash \Lambda a.v : \forall a.t_2$</td>
<td>rule CE-TAbs</td>
</tr>
<tr>
<td>$G \vdash (\Lambda a.v) \triangleright \forall a.\gamma : \forall a.t_1$</td>
<td>rule CE-Cast</td>
</tr>
</tbody>
</table>

**Rule CS-TBeta:** We have $e = (\Lambda a.v_1) t_2$ and $e' = v_1 t_2 / a$. We know $G \vdash \Lambda a.v_1 : \forall a.t_1$ (where our original type $t$ equals $t_1 t_2 / a$). Inversion on rule CE-TAbs gives us $G, a \vdash v_1 : t_1$. We can now use Lemma C.18 to get $G \vdash v_1 t_2 / a : t_1 t_2 / a$ as desired.

**Rule CS-TAppCong:** By the induction hypothesis.

**Rule CS-TAppPull:** We have $e = (v \triangleright \gamma) t_0$ where $G \vdash v : \forall a.t_2$, and we must prove $G \vdash v t_0 \triangleright (\gamma @ \langle t_0 \rangle) : t$.

<table>
<thead>
<tr>
<th>We know</th>
<th>How</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G \vdash (v \triangleright \gamma) t_0 : t$</td>
<td>assumption</td>
</tr>
<tr>
<td>$G \vdash v \triangleright \gamma : \forall a.t_1$</td>
<td>inversion of rule CE-TApp</td>
</tr>
<tr>
<td>$t = t_1 [t_0 / a]$</td>
<td>inversion of rule CE-TApp</td>
</tr>
<tr>
<td>$G \vdash \gamma : (\forall a.t_2) \sim (\forall a.t_1)$</td>
<td>inversion of rule CE-Cast</td>
</tr>
<tr>
<td>$G \vdash \langle t_0 \rangle : t_0 \sim t_0$</td>
<td>rule CG-Refl</td>
</tr>
<tr>
<td>$G \vdash \gamma @(t_0) : t_2 [t_0 / a] \sim t_1 [t_0 / a]$</td>
<td>rule CG-InstForAll</td>
</tr>
<tr>
<td>$G \vdash v t_0 : t_2 [t_0 / a]$</td>
<td>rule CE-TApp</td>
</tr>
<tr>
<td>$G \vdash v t_0 \triangleright (\gamma @(t_0)) : t_1 [t_0 / a]$</td>
<td>rule CE-Cast</td>
</tr>
</tbody>
</table>

**Rule CS-PackCong:** By the induction hypothesis.
Rule CS-OpenPack: Here, we have $e = \text{open} \, (\text{pack} \, t_1, v_0 \text{ as } t_0)$.

We know

- $G \vdash \text{open} \, (\text{pack} \, t_1, v_0 \text{ as } t_0) : t$
- $G \vdash \text{pack} \, t_1, v_0 \text{ as } t_0 : \exists \, a. t_2$

$$t = t_2[[\text{pack} \, t_1, v_0 \text{ as } t_0] / a]$$

- $G \vdash v_0 : t_2[t_1 / a]$

$$t_0 = \exists \, a. t_2$$

- $G \vdash t_0 : \text{type}$

- $G \vdash \langle t_0 \rangle : (\exists \, a. t_2) \sim (\exists \, a. t_2)$

- $G \vdash \text{projpack} \, t_1, v_0 \text{ as } t_0 : [\text{pack} \, t_1, v_0 \text{ as } t_0] \sim t_1$

- $G \vdash \text{sym} \, (\text{projpack} \, t_1, v_0 \text{ as } t_0) : t_2 \sim [\text{pack} \, t_1, v_0 \text{ as } t_0]$  

- $G \vdash \langle t_0 \rangle @(\text{sym} \, (\text{projpack} \, t_1, v_0 \text{ as } t_0)) : t_2[[\text{pack} \, t_1, v_0 \text{ as } t_0] / a] \sim t_2[[\text{pack} \, t_1, v_0 \text{ as } t_0] / a]$

- $G \vdash v_0 \triangleright \langle t_0 \rangle @(\text{sym} \, (\text{projpack} \, t_1, v_0 \text{ as } t_0)) : t_2[[\text{pack} \, t_1, v_0 \text{ as } t_0] / a] \sim t_2[[\text{pack} \, t_1, v_0 \text{ as } t_0] / a]$  

We thus see that the reduct has the same type as the redex, and we are done with this case.

Rule CS-OpenPackCasted: Similar to the previous case; note that we need rule CS-OpenPackCasted distinct from rule CS-OpenPack only to support determinism of reduction; otherwise both could be subsumed by a version of the rule that packed an expression $e$ instead of a value.

Rule CS-OpenCong: We must have $e = \text{open} \, e_0$. Inverting rule CE-Open in the derivation for $G \vdash \text{open} \, e_0 : t$ tells us $G \vdash e_0 : \exists \, a. t_2$ and $t = t_2[[e_0] / a]$. Given $G \vdash e_0 \rightarrow e'_0$, we must now show $G \vdash \text{open} \, e'_0 \triangleright (\exists \, a. t_2) @(\text{sym} \, \text{step} \, e) : t_2[[e_0] / a]$.  

We know

\[ G \vdash e_0 : \exists t. t \]
\[ G \vdash \text{step } e_0 : e_0 \sim e'_0 \]
\[ G \vdash [\text{step } e_0] : \{ e_0 \} \sim \{ e'_0 \} \]
\[ G \vdash \text{sym } [\text{step } e_0] : \{ e'_0 \} \sim \{ e_0 \} \]
\[ G \vdash \exists a. t_2 : \text{type} \]
\[ G \vdash (\exists a. t_2) \, @ (\text{sym } [\text{step } e_0]) : t_2[\{ e'_0 \} / a] \sim t_2[\{ e_0 \} / a] \]
\[ G \vdash \text{open } e'_0 : t_2[\{ e'_0 \} / a] \]
\[ G \vdash \text{open } e'_0 \, @ (\text{sym } [\text{step } e_0]) : t_2[\{ e_0 \} / a] \]

We are done with this case.

**Rule CS-OpenPull**: We have \( e = \text{open } (v \triangleright \gamma) \), where \( v = \text{pack } t_0, v_0 \) as \( \exists a. t_1 \).

We know

\[ G \vdash \text{open } (v \triangleright \gamma) : t \]
\[ G \vdash v \triangleright \gamma : \exists a. t_2 \]
\[ t = t_2[\{ v \triangleright \gamma \} / a] \]
\[ G \vdash v : t_3 \]
\[ t_3 = \exists a. t_1 \]
\[ G \vdash \gamma : (\exists a. t_1) \sim (\exists a. t_2) \]
\[ G \vdash v \triangleright \gamma : v \sim v \triangleright \gamma \]
\[ G \vdash [v \triangleright \gamma] : [v] \sim [v \triangleright \gamma] \]
\[ G \vdash \gamma @ [v \triangleright \gamma] : t_1[\{ v \} / a] \sim t_2[\{ v \triangleright \gamma \} / a] \]
\[ G \vdash \text{open } v : t_1[\{ v \} / a] \]
\[ G \vdash \text{open } v \, @ (v \triangleright \gamma) : t_2[\{ v \triangleright \gamma \} / a] \]

**Rule CS-Let**: We have \( e = \text{let } x = e_1 \text{ in } e_2. \)

We know

\[ G \vdash \text{let } x = e_1 \text{ in } e_2 : t \]
\[ G \vdash e_1 : t_1 \]
\[ G, x : t_1 \vdash e_2 : t_2 \]
\[ t = t_2[e_1 / x] \]
\[ G \vdash e_2[e_1 / x] : t_2[e_1 / x] \]

**Rule CS-CastCong**: We have \( e = e_0 \triangleright \gamma \), where \( G \vdash e_0 \rightarrow e'_0 \). We must show \( G \vdash e'_0 \triangleright \gamma : t. \)

We know

\[ G \vdash e_0 : t_0 \]
\[ G \vdash \gamma : t_0 \sim t \]
\[ G \vdash e'_0 : t_0 \]
\[ G \vdash e'_0 \triangleright \gamma : t \]

**Rule CS-CastTrans**: We have \( e = (v \triangleright \gamma_1) \triangleright \gamma_2 \), and we must prove \( G \vdash v \triangleright (\gamma_1 ; \gamma_2) : t. \)

We know

\[ G \vdash v \triangleright \gamma_1 : t_1 \]
\[ G \vdash \gamma_2 : t_1 \sim t \]
\[ G \vdash v : t_2 \]
\[ G \vdash \gamma_1 : t_2 \sim t \]
\[ G \vdash \gamma_1 ; \gamma_2 : t_2 \sim t \]
\[ G \vdash v \triangleright (\gamma_1 ; \gamma_2) : t \]

\( \square \)
C.4 Progress

Definition C.21 (Rewrite relation). Define rewrite relations on types \( t_1 \Rightarrow t_2 \) and terms \( e_1 \Rightarrow e_2 \) with the rules below.

\[
\begin{align*}
\text{RT-Refl} & : t \Rightarrow t \\
\text{RT-Base} & : t \Rightarrow t' \\
\text{RT-ForAll} & : \forall a.t \Rightarrow \forall a.t' \\
\text{RT-Exists} & : \exists a.t \Rightarrow \exists a.t' \\
\text{RT-App} & : e \Rightarrow e' \\
\text{RE-Pack} & : (\text{pack } t, e \text{ as } \exists a.t_0) \Rightarrow t' \\
\text{RE-App} & : e_1 \Rightarrow e'_1, e_2 \Rightarrow e'_2 \\
\text{RE-Open} & : e \Rightarrow e' \\
\end{align*}
\]

(rewriting relation on types)

\[
\begin{align*}
\text{RE-Ref} & : e \Rightarrow e' \\
\text{RE-DropCo} & : e \Rightarrow e' \\
\text{RE-AddCo} & : e \Rightarrow e' \\
\text{RE-Abs} & : e \Rightarrow e' \\
\end{align*}
\]

(rewriting relation on terms)

Lemma C.22. Define \( \Rightarrow^* \) to be the reflexive, transitive closure of \( \Rightarrow \).

Lemma C.23 (Type substitution in rewrite relation).
1. If \( t_1 \Rightarrow t_2 \), then \( t_1[t_3/a] \Rightarrow t_2[t_3/a] \).
2. If \( e_1 \Rightarrow e_2 \), then \( e_1[t_3/a] \Rightarrow e_2[t_3/a] \).

Proof. By mutual induction on the structure of \( t_1 \Rightarrow t_2 \) or \( e_1 \Rightarrow e_2 \).

Lemma C.24 (Type substitution in transitive rewrite relation).
1. If \( t_1 \Rightarrow^* t_2 \), then \( t_1[t_3/a] \Rightarrow^* t_2[t_3/a] \).
2. If \( e_1 \Rightarrow^* e_2 \), then \( e_1[t_3/a] \Rightarrow^* e_2[t_3/a] \).

Proof. By induction on the length of the reduction.

Lemma C.25 (Substitution in rewrite relation).
1. If \( t_1 \Rightarrow t_2 \), then \( t_1[e_3/x] \Rightarrow t_2[e_3/x] \).
2. If \( e_1 \Rightarrow e_2 \), then \( e_1[e_3/x] \Rightarrow e_2[e_3/x] \).

Proof. By mutual induction on the structure of \( t_1 \Rightarrow t_2 \) or \( e_1 \Rightarrow e_2 \).
Lemma C.26 (Substitution in the transitive rewrite relation).

(1) If $t_1 \Rightarrow^* t_2$, then $t_1[e_3/x] \Rightarrow^* t_2[e_3/x]$.

(2) If $e_1 \Rightarrow^* e_2$, then $e_1[e_3/x] \Rightarrow^* e_2[e_3/x]$.

Proof. By induction on the length of the reduction. □

Lemma C.27 (Lifting in transitive rewrite relation). Assume $t_1 \Rightarrow t_2$.

(1) For every $t_3$, $t_3[t_1/a] \Rightarrow t_3[t_2/a]$.

(2) For every $e_3$, $e_3[t_1/a] \Rightarrow e_3[t_2/a]$.

Proof. By mutual induction on the structure of $t_3$ and $e_3$.

$t_3 = a'$: We have two cases:

$a' = a$: We are done by assumption.

$a' \neq a$: We are done by rule RT-REFL.

$t_3 = B\tilde{t}$: By the induction hypothesis and rule RT-BASE.

$t_3 = \forall \alpha. t_4$: By the induction hypothesis and rule RT-FORALL.

$t_3 = \exists \alpha'. t_4$: By the induction hypothesis and rule RT-EXISTS.

$t_3 = [e]$: By the induction hypothesis and rule RT-PROJ.

$e_3 = x$: By rule RE-REFL.

$e_3 = \lambda x : t. e$: By the induction hypothesis and rule RE-ABS.

$e_3 = e_1 e_2$: By the induction hypothesis and rule RE-APP.

$e_3 = \Lambda a. e$: By the induction hypothesis and rule RE-TABS.

$e_3 = e t$: By the induction hypothesis and rule RE-TAPP.

$e_3 = \text{pack } t, e \text{ as } t'$: By the induction hypothesis and rule RE-PACK.

$e_3 = \text{open } e$: By the induction hypothesis and rule RE-OPEN.

$e_3 = \text{let } x = e_1 \text{ in } e_2$: By the induction hypothesis and rule RE-LETCONG.

$e_3 = e_3 \triangleright y$: By the induction hypothesis and rule RE-CAST. Note that the resulting coercion need not be related to the initial coercion. □

Lemma C.28 (Lifting in transitive rewrite relation). Assume $t_1 \Rightarrow^* t_2$.

(1) For every $t_3$, $t_3[t_1/a] \Rightarrow^* t_3[t_2/a]$.

(2) For every $e_3$, $e_3[t_1/a] \Rightarrow^* e_3[t_2/a]$.

Proof. By induction on the length of the reduction. □

Lemma C.29 (Parallel substitution of a type). Assume $t_1 \Rightarrow t_2$.

(1) If $t_3 \Rightarrow t_4$, then $t_3[t_1/a] \Rightarrow t_4[t_2/a]$.

(2) If $e_3 \Rightarrow e_4$, then $e_3[t_1/a] \Rightarrow e_4[t_2/a]$.

Proof. By mutual induction on $t_3 \Rightarrow t_4$ or $e_3 \Rightarrow e_4$.

Rule RT-REFL: By Lemma C.27.

Rule RT-BASE: By the induction hypothesis.

Rule RT-FORALL: By the induction hypothesis.

Rule RT-EXISTS: By the induction hypothesis.

Rule RT-PROJ: By the induction hypothesis.

Rule RT-PROJPACK: By the induction hypothesis.

Rule RE-REFL: By Lemma C.27.

Rule RE-DROPCO: By the induction hypothesis.

Rule RE-ADDCO: By the induction hypothesis.

Rule RE-ABS: By the induction hypothesis.
Rule RE-APP: By the induction hypothesis.
Rule RE-TABS: By the induction hypothesis.
Rule RE-TAPP: By the induction hypothesis.
Rule RE-PACK: By the induction hypothesis.
Rule RE-OPEN: By the induction hypothesis.
Rule RE-LETCONG: By the induction hypothesis.
Rule RE-CAST: By the induction hypothesis.
Rule RE-BETA: By the induction hypothesis.
Rule RE-TBETA: By the induction hypothesis, noting that the bound variable in the rule can be considered distinct from the variable being substituted.
Rule RE-OPENPACK: By the induction hypothesis.
Rule RE-LET: By the induction hypothesis.

Lemma C.30 (Parallel Substitution). Assume $e_1 \Rightarrow e_2$.

1. If $t_3 \Rightarrow t_4$, then $t_3[e_1 / x] \Rightarrow t_4[e_2 / x]$.
2. If $e_3 \Rightarrow e_4$, then $e_3[e_1 / x] \Rightarrow e_4[e_2 / x]$.

Proof. Similar to previous proof.

Lemma C.31 (Local Diamond).

1. If $t_1 \Rightarrow t_2$ and $t_1 \Rightarrow t_3$, then there exists $t_4$ such that $t_2 \Rightarrow t_4$ and $t_3 \Rightarrow t_4$.
2. If $e_1 \Rightarrow e_2$ and $e_1 \Rightarrow e_3$, then there exists $e_4$ such that $e_2 \Rightarrow e_4$ and $e_3 \Rightarrow e_4$.

Proof. By mutual induction on the derivation for $t_1 \Rightarrow t_2$ or $e_1 \Rightarrow e_2$. In all cases, if $t_1 \Rightarrow t_3$ or $e_1 \Rightarrow e_3$ is by rule RT-REFL or rule RE-REFL, then we are done, with the common reduct being $t_2$ or $e_2$. We thus ignore the possibility that $t_1 \Rightarrow t_3$ can be by rule RT-REFL or that $e_1 \Rightarrow e_3$ can be by rule RE-REFL. Similarly, the use of rule RE-ADDCo to rewrite $e_1 \Rightarrow e_3$ can be countered by a use of rule RE-DROPCo in $e_3 \Rightarrow e_4$, keeping the rest of the case untouched; we thus ignore the possibility of rule RE-ADDCo for $e_1 \Rightarrow e_3$.

Rule RT-REFL: In this case, $t_2 = t_1$ and $t_3$ can be the common reduct.

Rule RT-BASE: The rewrite $t_1 \Rightarrow t_3$ must also be by rule RT-BASE. We are done by applying the induction hypothesis.

Rule RT-FORALL: The rewrite $t_1 \Rightarrow t_3$ must also be by rule RT-FORALL. We are done by applying the induction hypothesis.

Rule RT-EXISTS: The rewrite $t_1 \Rightarrow t_3$ must also be by rule RT-EXISTS. We are done by applying the induction hypothesis.

Rule RT-PROJ: We have two cases, depending on how $t_1 \Rightarrow t_3$ was rewritten:

Rule RT-PROJ: By the induction hypothesis.

Rule RT-PROJPACK: We have $t_1 = [\text{pack } t, e \text{ as } \exists a.t_0]$ and $t_2 = [e'_0]$, where pack $t, e \text{ as } \exists a.t_0 \Rightarrow e'_0$. We further have $t_3 = t'$ where $t \Rightarrow t'$.

We know $e'_0 = \text{pack } t'', e'' \text{ as } \exists a.t_0''$ and $t \Rightarrow t''$ such that $t' \Rightarrow t''$ and $t'' \Rightarrow t'''$.

We also know $t_4 = t'''$ and $t_2 \Rightarrow t'''$.

Rule RT-PROJPACK: We have two cases, depending on how $t_1 \Rightarrow t_3$ was rewritten:

Rule RT-PROJ: Like the rule RT-PROJ rule RT-PROJPACK case above.
Rule RT-ProjPack: We are done by the induction hypothesis.
Rule RE-Refl: In this case, \( e_2 = e_1 \) and \( e_3 \) can be the common reduct.
Rule RE-DropCo: We have two cases, depending on how \( e_1 \Rightarrow e_3 \) was rewritten:
  - Rule RE-Cast: In this case, \( e_1 = e \Rightarrow e' \Rightarrow \gamma \) and \( e_3 = e' \Rightarrow \gamma' \) where \( e \Rightarrow e' \). The induction hypothesis gives us \( e_0 \) such that \( e_2 \Rightarrow e_0 \) and \( e' \Rightarrow e_0 \). Choose \( e_4 = e_0 \). We see that \( e_2 \Rightarrow e_4 \) (from the induction hypothesis) and \( e_3 \Rightarrow e_4 \) by rule RE-Coherence.
  - Rule RE-AddCo: In this case, \( e_2 = e' \Rightarrow \gamma \) where \( e_1 \Rightarrow e' \). Use the induction hypothesis to get \( e_3 \) such that \( e' \Rightarrow e_5 \) and \( e_3 \Rightarrow e_5 \). Choose \( e_4 = e_5 \). We conclude that \( e_2 \Rightarrow e_4 \) by rule RE-DropCo.
Rule RE-Abs: By the induction hypothesis.
Rule RE-App: We have two cases, depending on how \( e_1 \Rightarrow e_3 \) was rewritten:
  - Rule RE-App: By the induction hypothesis.
  - Rule RE-Beta: We have \( e_1 = (\lambda x: t_1. e_5) \ e_6 \), \( e_2 = (\lambda x: t_2. e_7) \ e_8 \) (where \( t_1 \Rightarrow t_2 \), \( e_5 \Rightarrow e_7 \), and \( e_6 \Rightarrow e_8 \) (inverting rule RE-Abs)), and \( e_3 = e_0 [e_{10} / x] \) (where \( e_5 \Rightarrow e_9 \) and \( e_6 \Rightarrow e_{10} \)).

\[
\begin{array}{l|l}
\text{We know} & \text{How} \\
\hline
\text{e}_{11} \text{ such that } e_7 \Rightarrow e_{11} \text{ and } e_9 \Rightarrow e_{11} & \text{induction hypothesis} \\
\text{e}_{12} \text{ such that } e_8 \Rightarrow e_{12} \text{ and } e_{10} \Rightarrow e_{12} & \text{induction hypothesis} \\
\text{Choose } e_4 = e_{11} [e_{12} / x] & \text{rule RE-Beta} \\
\text{e}_2 \Rightarrow e_4 & \text{Lemma C.30} \\
\text{e}_3 \Rightarrow e_4 & \text{Lemma C.30}
\end{array}
\]
Rule RE-Tabs: By the induction hypothesis.
Rule RE-Tapp: Similar to the rule RE-App case, but referring to rule RE-TBeta and Lemma C.29.
Rule RE-Pack: By the induction hypothesis.
Rule RE-Open: Similar to the rule RE-DropCo case, but referring to rule RE-OpenPack.
Rule LetCong: Similar to the rule RE-App case, but referring to rule RE-Let. This case uses Lemma C.30.
Rule Cast: By the induction hypothesis or following the logic in the case for rules RE-DropCo and RE-Cast.
Rule Beta: We have two cases, depending on how \( e_1 \Rightarrow e_3 \) was rewritten.
  - Rule RE-Beta: We have \( e_1 = (\lambda x: t_1. e_5) \ e_6 \), \( e_2 = e_7 [e_8 / x] \) (where \( e_5 \Rightarrow e_7 \) and \( e_6 \Rightarrow e_8 \)), and \( e_3 = e_9 [e_{10} / x] \) (where \( e_5 \Rightarrow e_9 \) and \( e_6 \Rightarrow e_{10} \)).

\[
\begin{array}{l|l}
\text{We know} & \text{How} \\
\hline
\text{e}_{11} \text{ such that } e_7 \Rightarrow e_{11} \text{ and } e_9 \Rightarrow e_{11} & \text{induction hypothesis} \\
\text{e}_{12} \text{ such that } e_8 \Rightarrow e_{12} \text{ and } e_{10} \Rightarrow e_{12} & \text{induction hypothesis} \\
\text{Choose } e_4 = e_{11} [e_{12} / x] & \text{Lemma C.30} \\
\text{e}_2 \Rightarrow e_4 & \text{Lemma C.30} \\
\text{e}_3 \Rightarrow e_4 & \text{Lemma C.30}
\end{array}
\]
Rule RE-TBeta: Like the case for rule RE-Beta, but referring to rule RE-Tapp and Lemma C.29.
Rule RE-OpenPack: By the induction hypothesis or following the logic in the case for rules RE-Open and RE-OpenPack.
Rule RE-Let: Like the case for rule RE-Beta, but referring to rule RE-LetCong. This case uses Lemma C.30.

□
Lemma C.32 (Confluence). If \( t_1 \Rightarrow^* t_2 \) and \( t_1 \Rightarrow t_3 \), then there exists \( t_4 \) such that \( t_2 \Rightarrow^* t_4 \) and \( t_3 \Rightarrow^* t_4 \).

Proof. Corollary of Lemma C.31. (See e.g. Baader and Nipkow [1998, Lemma 2.7.4].) \( \square \)

Lemma C.33 (Rewriting existentials). If \( \exists a.t_1 \Rightarrow^* t_3 \) and \( \exists a.t_2 \Rightarrow^* t_3 \), then there exists \( t_4 \) such that \( t_1 \Rightarrow^* t_4 \) and \( t_2 \Rightarrow^* t_4 \).

Proof. Ignoring reflexivity, the only rule that applies to \( \exists a.t_1 \) and \( \exists a.t_2 \) is rule RT-EXISTS. Accordingly, an inductive argument shows that \( t_3 \) must have the form \( \exists a.t_4 \) for some \( t_4 \). Furthermore, the argument that reveals \( t_4 \) also shows that \( t_1 \Rightarrow^* t_4 \) and \( t_2 \Rightarrow^* t_4 \) as desired. \( \square \)

Lemma C.34 (Rewriting existentials). If \( \forall a.t_1 \Rightarrow^* t_3 \) and \( \forall a.t_2 \Rightarrow^* t_3 \), then there exists \( t_4 \) such that \( t_1 \Rightarrow^* t_4 \) and \( t_2 \Rightarrow^* t_4 \).

Proof. Similar to proof of Lemma C.33. \( \square \)

Lemma C.35 (Rewriting base types). If \( B \bar{t} \Rightarrow^* t_0 \) and \( B \bar{t}' \Rightarrow^* t_0 \), then, for each \( i \), there exists \( t_i'' \) such that \( t_i \Rightarrow^* t_i'' \) and \( t_i' \Rightarrow t_i'' \).

Proof. Similar to proof of Lemma C.33. \( \square \)

Lemma C.36 (Rewriting subsumes reduction). If \( G \vdash e_1 \rightarrow e_2 \), then \( e_1 \Rightarrow e_2 \).

Proof. By induction on the structure of \( G \vdash e_1 \rightarrow e_2 \). (We leave out uses of rule RE-REFL throughout.)

Rule CS-BETA: By rule RE-BETA.

Rule CS-APPCONG: By the induction hypothesis and rule RE-APP.

Rule CS-APP_PULL: By rules RE-ADDCo, RE-APP, RE-DROPCo, and RE-ADDCo.

Rule CS-TABSCONG: By the induction hypothesis and rule RE-TABS.

Rule CS-TABS_PULL: By rules RE-ADDCo, RE-TABS, and RE-DROPCo.

Rule CS-TBETA: By rule RE-TBETA.

Rule CS-TAPPCONG: By the induction hypothesis and rule RE-TAPP.

Rule CS-TAPP_PULL: By rules RE-ADDCo, RE-TAPP, and RE-DROPCo.

Rule CS-PACKCONG: By the induction hypothesis and rule RE-PACK.

Rule CS-OPENPACK: By rules RE-OPENPACK and RE-ADDCo.

Rule CS-OPENPACKCASTED: By rules RE-OPENPACK and RE-ADDCo.

Rule CS-OPENCONG: By the induction hypothesis and rule RE-OPEN.


Rule CS-LET: By rule RE-LET.

Rule CS-CASTCONG: By the induction hypothesis and rule RE-CAST.

Rule CS-CASTTRANS: by rules RE-CAST and RE-DROPCo.

Lemma C.37 (Completeness of the rewrite relation). If \( G \vdash \gamma : t_1 \sim t_2 \), then there exists \( t_3 \) such that \( t_1 \Rightarrow^* t_3 \) and \( t_2 \Rightarrow^* t_3 \).

Proof. By induction on the structure of the typing judgment.

Rule CG-REFL: Trivial.

Rule CG-SYM: By the induction hypothesis.
Rule CG-TRANS: We have $\gamma = \gamma_1 \vdash \gamma_2$.

We know

\[ G \vdash \gamma_1 : t_1 \sim t_4 \]
\[ G \vdash \gamma_2 : t_4 \sim t_2 \]
\[ t_5 \text{ such that } t_1 \Rightarrow^* t_5 \text{ and } t_4 \Rightarrow^* t_5 \]
\[ t_6 \text{ such that } t_4 \Rightarrow^* t_6 \text{ and } t_2 \Rightarrow^* t_6 \]
\[ t_7 \text{ such that } t_5 \Rightarrow^* t_7 \text{ and } t_6 \Rightarrow^* t_7 \]

How

inversion of rule CG-TRANS
inversion of rule CG-TRANS
induction hypothesis
induction hypothesis
Lemma C.32

We are done, as $t_1 \Rightarrow^* t_7$ and $t_2 \Rightarrow^* t_7$.

Rule CG-BASE: By the induction hypothesis and rule RT-BASE.

Rule CG-FORALL: By the induction hypothesis and rule RT-FORALL.

Rule CG-EXISTS: By the induction hypothesis and rule RT-EXISTS.

Rule CG-PROJ: We have $\gamma = [\eta]$, where $G \vdash \eta : e_1 \sim e_2$. We must show that $[e_1]$ and $[e_2]$ are joinable. We have two cases, depending on the rule used to prove $G \vdash \eta : e_1 \sim e_2$.

Rule CH-COHERENCE: In this case, $e_2 = e_1 \triangleright^\gamma$. The common reduct is $[e_1]$, and we are done by rule RE-DropCo.

Rule CH-STEP: In this case, $G \vdash e_1 \rightarrow e_2$. Lemma C.36 tells us $e_1 \Rightarrow e_2$; we are done by rule RE-PROJ.

Rule CG-PROJPACK: We are done by rule RT-PROJPACK and rule RT-REFL.

Rule CG-INSTFORALL: Similar to the case below, but using Lemma C.34.

Rule CG-INSTEXISTS: We have $\gamma = \gamma_1 \uplus \gamma_2$.

We know

\[ G \vdash \gamma_1 : (\exists \ a.t_4) \sim (\exists \ a.t_5) \]
\[ G \vdash \gamma_2 : t_6 \sim t_7 \]
\[ t_8 \text{ that is the join of } \exists \ a.t_4 \text{ and } \exists \ a.t_5 \]
\[ t_9 \text{ that is the join of } t_6 \text{ and } t_7 \]
\[ t_{10} \text{ that is the join of } t_4 \text{ and } t_5 \]
\[ t_{10}[t_6 / a] \Rightarrow^* t_{10}[t_6 / a] \]
\[ t_{10}[t_7 / a] \Rightarrow^* t_{10}[t_7 / a] \]
\[ t_{10}[t_6 / a] \Rightarrow^* t_{10}[t_9 / a] \]
\[ t_{10}[t_7 / a] \Rightarrow^* t_{10}[t_9 / a] \]
\[ t_{10}[t_6 / a] \text{ is the join of } t_4[t_6 / a] \text{ and } t_5[t_7 / a] \]

How

inversion of rule CG-INSTEXISTS
inversion of rule CG-INSTEXISTS
induction hypothesis
induction hypothesis
Lemma C.33
Lemma C.24
Lemma C.24
Lemma C.28
Lemma C.28
transitivity

Rule CG-NTH: By the induction hypothesis and Lemma C.35.

\[ 
\]

Definition C.38 (Value type). If $t$ is a value type, then $t$ is one of the following:

1. a base type $B^\gamma$
2. a universal type $\forall \ a.t'$
3. an existential type $\exists \ a.t'$

Definition C.39 (Type head). If $t$ is a value type, then define head$(t)$ by the following equations:

\[ 
\text{head}(B^\gamma) = B \\
\text{head}(\forall \ a.t) = \forall \\
\text{head}(\exists \ a.t) = \exists \\
\]

Lemma C.40 (Value types). If $G \vdash v : t$, then $t$ is a value type.

Proof. Straightforward case analysis on the structure of $v$. □
**Lemma C.41 (Preservation of Value Types).** If $t$ is a value type and $t \Rightarrow^* t'$, then $t'$ is a value type and $\text{head}(t) = \text{head}(t')$. 

**Proof.** By induction over the length of the chain $t \Rightarrow^* t'$. 

**Zero steps:** Trivial. 

**$n + 1$ steps:** We have $t_0$ such that $t \Rightarrow^* t_0$ in $n$ steps and that $t_0 \Rightarrow t'$. The induction hypothesis tells us that $t_0$ is a value type and that $\text{head}(t) = \text{head}(t_0)$. Analyzing how $t_0$ rewrites to $t'$, we see it must be by rule RT-Base, rule RT-ForAll, or rule RT-Exists. In any of these cases $t'$ is a value type such that $\text{head}(t_0) = \text{head}(t')$. 

**Lemma C.42 (Consistency).** If $G \vdash \gamma : t_1 \sim t_2$ and both $t_1$ and $t_2$ are value types, then $\text{head}(t_1) = \text{head}(t_2)$. 

**Proof.** Lemma C.37 gives us $t_3$ such that $t_1 \Rightarrow^* t_3$ and $t_2 \Rightarrow^* t_3$. Lemma C.41 then tells us that $t_3$ is a value type with $\text{head}(t_3) = \text{head}(t_1)$. Another use of Lemma C.41 tells us that $\text{head}(t_3) = \text{head}(t_2)$. By transitivity of equality, $\text{head}(t_1) = \text{head}(t_2)$. 

**Lemma C.43 (Canonical Forms).** 

1. If $G \vdash v : t_1 \rightarrow t_2$, then there exist $x$ and $e$ such that $v = \lambda x : t_1 . e$.
2. If $G \vdash v : \forall a . t$, then there exists $v_0$ such that $v = \Lambda a . v_0$.
3. If $G \vdash v : \exists a . t$, then either:
   a. there exists $t_0, v_0, \text{and } t_1$ such that $v = \text{pack } t_0, v_0 \text{ as } t_1$, or
   b. there exists $t_0, v_0, v_0, \text{and } t_1$ such that $v = \text{pack } t_0, (v_0 \triangleright v_0) \text{ as } t_1$

**Proof.** 


**Theorem C.44 (Progress).** If $G \vdash e : t$, where $G$ contains only type variable bindings, then one of the following is true: 

1. There exists $e'$ such that $G \vdash e \rightarrow e'$; 
2. $e$ is a value $v$; or 
3. $e$ is a casted value $v \triangleright \gamma$.

**Proof.** By induction on the structure of the typing judgment. 

**Rule CE-VAR:** Impossible, as $G$ contains only type variable bindings. 

**Rule CE-INt:** Here, $e = n$, a value. 

**Rule CE-ABS:** Here, $e = \lambda x : t_1 . e_1$, a value. 

**Rule CE-APP:** We know $e = e_1 e_2$, with $G \vdash e_1 : t_1 \rightarrow t_2$ and $G \vdash e_2 : t_1$. Applying the induction hypothesis on the first of these yields three possibilities: 

- **There exists $e_1'$ such that** $G \vdash e_1 \rightarrow e_1'$: In this case, $e_1 e_2$ steps by rule CS-AppCong. 
  
- $e_1 = v_1$: Lemma C.43 tells us that $v_1 = \lambda x : t_1 . e_0$. Thus, our original expression is $e = (\lambda x : t_1 . e_0) e_2$, which can reduce by rule CS-Beta. 
  
- $e_1 = v_1 \triangleright v_1$: Thus, our original expression is $e = (v_1 \triangleright v_1) e_2$. In order to use rule CS-AppPull, we need only prove $v_1 = \lambda x : t_3 . e_0$ for some $t_3$ and $e_0$. 

We know

<table>
<thead>
<tr>
<th>How</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G \vdash (v_1 \triangleright y_1) \ e_2 : t )</td>
</tr>
<tr>
<td>( G \vdash v_1 \triangleright y_1 : t_4 \rightarrow t )</td>
</tr>
<tr>
<td>( G \vdash v_1 : t_5 )</td>
</tr>
<tr>
<td>( G \vdash y_1 : t_5 \sim (t_4 \rightarrow t) )</td>
</tr>
<tr>
<td>( t_5 ) is a value type</td>
</tr>
<tr>
<td>( t_5 = t_6 \rightarrow t_7 )</td>
</tr>
<tr>
<td>( v_1 = \lambda x : t_3 . e_0 )</td>
</tr>
</tbody>
</table>

We can thus use rule CS-APP\textsc{Pull}, and we are done with this case.

**Rule CE-TABS:** Here, \( e = \Lambda a. e_0 \), where \( G, a \vdash e_0 : t_0 \) and \( t = \forall a. t_0 \). Using the induction hypothesis on \( e_0 \) gives us three possibilities:

**There exists** \( e'_0 \) such that \( G, a \vdash e_0 \rightarrow e'_0 \): We are done by rule CS-TABS\textsc{Cong}.

\( e_0 = v_0 \): The expression \( e = \Lambda a. v_0 \) is a value.

\( e_0 = v_0 \triangleright y_0 \): We are done by rule CS-TABS\textsc{Pull}.

**Rule CE-TAPP:** We know \( e = e_0 t_0 \), with \( G \vdash e_0 : \forall a. t_1 \) and \( G \vdash t_0 : \text{type} \). A use of the induction hypothesis on \( e_0 \) yields three cases:

**There exists** \( e'_0 \) such that \( G \vdash e_0 \rightarrow e'_0 \): We are done by rule CS-TAPP\textsc{Cong}.

\( e_0 = v_0 \): We have \( e = v_0 t_0 \). Lemma C.43 tells us that \( v_0 = \Lambda a. v_1 \), and thus that \( e = (\Lambda a. v_1) t_0 \). We are done by rule CS-TBETA.

\( e_0 = v_0 \triangleright y_0 \): We have \( e = (v_0 \triangleright y_0) t_0 \). To use rule CS-TAPP\textsc{Pull}, we must show \( G \vdash v_0 : \forall a. t_1 \).

We know

<table>
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<tr>
<th>How</th>
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<tbody>
<tr>
<td>( G \vdash (v_0 \triangleright y_0) t_0 : t )</td>
</tr>
<tr>
<td>( G \vdash v_0 \triangleright y_0 : \forall a. t_3 )</td>
</tr>
<tr>
<td>( G \vdash v_0 : t_4 )</td>
</tr>
<tr>
<td>( G \vdash y_0 : t_4 \sim \forall a. t_3 )</td>
</tr>
<tr>
<td>( t_4 ) is a value type</td>
</tr>
<tr>
<td>( t_4 = \forall a. t_1 )</td>
</tr>
</tbody>
</table>

We can now use rule CS-TAPP\textsc{Pull}, and so we are done with this case.

**Rule CE-Pack:** We know \( e = \text{pack} \ t_0, e_0 \) as \( \exists a. t_1 \), where \( G \vdash e_0 : t_1 [t_0 / a] \). We use the induction hypothesis on \( e_0 \) to get three cases:

**There exists** \( e'_0 \) such that \( G \vdash e_0 \rightarrow e'_0 \): We are done by rule CS-Pack\textsc{Cong}.

\( e_0 = v_0 \) Then \( e = \text{pack} \ t_0, v_0 \) as \( \exists a. t_1 \) is a value.

\( e_0 = v_0 \triangleright y_0 \) In this case, we have \( e = \text{pack} \ t_0, (v_0 \triangleright y_0) \) as \( \exists a. t_1 \), which is a value.

**Rule CE-Open:** We know \( e = \text{open} \ e_0 \), where \( G \vdash e_0 : \exists a. t_0 \). Using the induction hypothesis on \( e_0 \) gives us three possibilities:

**There exists** \( e'_0 \) such that \( G \vdash e_0 \rightarrow e'_0 \): We are done by rule CS-Open\textsc{Cong}.

\( e_0 = v_0 \) Lemma C.43 gives us two cases, depending on whether the packed value is casted. If it is not, we are done by rule CS-Open\textsc{Pack}; if it is, we are done by rule CS-Open\textsc{PackCasted}.

\( e_0 = v_0 \triangleright y_0 \) In this case, we have \( e = \text{open} (v_0 \triangleright y_0) \). To use rule CS-Open\textsc{Pull}, we must show only that \( v_0 = \text{pack} t_1, v_1 \) as \( \exists a. t_0 \).
We know
\[ G \vdash \text{open} (v_0 \triangleright y_0) : t \] assumption
\[ G \vdash v_0 \triangleright y_0 : \exists a.t_2 \] inversion of rule CE-OPEN
\[ t = t_2[[v_0 \triangleright y_0] / a] \] inversion of rule CE-OPEN
\[ G \vdash v_0 : t_3 \] inversion of rule CE-CAST
\[ G \vdash y_0 : t_3 \sim \exists a.t_2 \] inversion of rule CE-CAST
\[ t_3 \] is a value type Lemma C.40
\[ t_3 = \exists a.t_4 \] Lemma C.42
\[ v_0 = \text{pack} t_1, v_1 \text{ as } \exists a.t_0 \] Lemma C.43
We are thus done by rule CS-OPENPULL.

Rule CE-LET: We are done by rule CS-LET.

Rule CE-CAST: We know \( e = e_0 \triangleright y_0 \), where \( G \vdash e_0 : t_0 \). We use the induction hypothesis on \( e_0 \) to get three cases:

There exists \( e'_0 \) such that \( G \vdash e_0 \rightarrow e'_0 \): We are done by rule CS-CASTCong.
\[ e_0 = v_0: \text{Then } e \text{ is a casted value } v_0 \triangleright y_0 \text{ and we are done.} \]
\[ e_0 = v_0 \triangleright y_1: \text{We are done by rule CS-CASTTRANS.} \]
\[ \square \]

C.5 Erasure
An erased expression \( \hat{e} \) is defined with the following grammar:
\[
\hat{e} ::= x \mid \lambda x.e \mid \hat{e}_1 \hat{e}_2 \mid \text{let } x = \hat{e}_1 \text{ in } \hat{e}_2 \mid n
\]
\[
\hat{o} ::= \lambda x.e \mid n
\]
Define the erasure function over core expressions with the following equations:
\[
|\hat{x}| = x
\]
\[
|\lambda x:t.e| = \lambda x.|e|
\]
\[
|\hat{e}_1 e_2| = |e_1| |e_2|
\]
\[
|\Lambda a.e| = |e|
\]
\[
|e t| = |e|
\]
\[
|\text{pack } t, e \text{ as } t_2| = |e|
\]
\[
|\text{open } e| = |e|
\]
\[
|\text{let } x = e_1 \text{ in } e_2| = |\text{let } x = |e_1| \text{ in } |e_2|
\]
\[
|e \triangleright y| = |e|
\]
\[
|n| = n
\]
The single-step operational semantics of erased expressions is given by these rules:

\[ \hat{e} \rightarrow \hat{e}' \] (Single-step operational semantics)

\[
\begin{array}{ccc}
\text{ES-BETA} & \text{ES-APP} & \text{ES-LET} \\
(\lambda x.\hat{e}_1) \hat{e}_2 & \hat{e}_1 \rightarrow \hat{e}_1'[\hat{e}_2 / x] & \hat{e}_1 \hat{e}_2 & \rightarrow \hat{e}_1'[\hat{e}_2 / x] & \text{let } x = \hat{e}_1 \text{ in } \hat{e}_2 & \rightarrow \hat{e}_1'[\hat{e}_2 / x] \\
\end{array}
\]

Lemma C.45 (Erasure Substitution). For all expressions \( e_1 \) and \( e_2 \), \( |e_1[e_2 / x]| = |e_1|[]|e_2| / x| \).

Proof. Straightforward induction on the structure of \( e_1 \). \( \square \)
Lemma C.46 (Erasure type substitution). For all expressions $e$ and types $t$, $|e[t/a]| = |e|$.

Proof. Straightforward induction on the structure of $e$. \hfill \Box

Lemma C.47 (Single-step erasure ($\Rightarrow$)). If $G \vdash e \rightarrow e'$, then either $|e| = |e'|$ or $|e| \rightarrow |e'|$.

Proof. By induction on the structure of $G \vdash e \rightarrow e'$.

Rule CS-BETA: By rule ES-BETA and Lemma C.45.

Rule CS-APP: By the induction hypothesis and rule ES-APP.

Rule CS-APP: Here, $|e| = |e'|$.

Rule CS-TABS: By the induction hypothesis.

Rule CS-TABS: Here, $|e| = |e'|$.

Rule CS-TABS: By Lemma C.46.

Rule CS-TABS: By the induction hypothesis.

Rule CS-TAPP: Here, $|e| = |e'|$.

Rule CS-TAPP: By the induction hypothesis.

Rule CS-PACK: Here, $|e| = |e'|$.

Rule CS-PACK: By the induction hypothesis.

Rule CS-PACK: Here, $|e| = |e'|$.

Rule CS-PACKCASTED: By the induction hypothesis.

Rule CS-PACKCASTED: Here, $|e| = |e'|$.

Rule CS-PACKCASTED: By the induction hypothesis.

Rule CS-PACKCASTED: Here, $|e| = |e'|$.

Rule CS-PACKCASTED: By the induction hypothesis.

Rule CS-PACKCASTED: Here, $|e| = |e'|$.

Rule CS-CAST: By rule ES-LET and Lemma C.45.

Rule CS-CAST: By the induction hypothesis.

Rule CS-CAST: Here, $|e| = |e'|$.

\hfill \Box

Theorem C.48 (Erasure). If $G \vdash e \rightarrow^* e'$, then $|e| \rightarrow^* |e'|$.

Proof. By induction on the length of the reduction, appealing to Lemma C.47. \hfill \Box